

Topological Free Entropy Dimensions in Nuclear C^* -algebras and in Full Free Products of Unital C^* -algebras

Don Hadwin

Qihui Li

and

Junhao Shen¹

Abstract: In the paper, we introduce a new concept, topological orbit dimension, of n -tuple of elements in a unital C^* -algebra. Using this concept, we conclude that Voiculescu's topological free entropy dimension of every finite family of self-adjoint generators of a nuclear C^* -algebra is less than or equal to 1. We also show that the Voiculescu's topological free entropy dimension is additive in the full free product of some unital C^* -algebras. In the appendix, we show that unital full free product of Blackadar and Kirchberg's unital MF algebras is also MF algebra. As an application, we obtain that $Ext(C_r^*(F_2) *_\mathbb{C} C_r^*(F_2))$ is not a group.

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1. Introduction

The theory of free probability and free entropy was developed by Voiculescu from 1980s. His theory plays a crucial role in the recent study of finite von Neumann algebras (see [5], [8], [10], [11], [12], [13], [14], [17], [23], [24], [34], [35], [36]). The notion of topological free entropy dimension of n -tuple of elements in a unital C^* -algebra, as an analogue of free entropy dimension for finite von Neumann algebras in C^* -algebra context, was also introduced by Voiculescu in [37], where basic properties of free entropy dimension are discussed.

We started our investigation on properties of topological free entropy dimension in [18], where we computed the topological free entropy dimension of a self-adjoint element in a unital C^* -algebra. Some estimations of topological free entropy dimensions in infinite dimensional, unital, simple C^* -algebras with a unique trace, which include irrational rotation C^* -algebra, UHF algebra and $C_{red}^*(F_2) \otimes_{min} C_{red}^*(F_2)$, were also obtained in the same paper. In [19], we proved a formula of topological free entropy dimension in the orthogonal sum (or direct sum) of unital C^* -algebras. As a corollary, we computed the topological free entropy dimension of every finite family of self-adjoint generators of a finite dimensional C^* -algebra. In this article, we will continue our investigation on the concept of Voiculescu's topological free entropy dimension.

To study Voiculescu's topological free entropy dimension, firstly we introduce a notion of topological orbit dimension $\mathfrak{K}_{top}^{(2)}$, a modification of "topological free orbit dimension" in [18] which is inspired by the paper [17], of n -tuple of self-adjoint elements in a unital C^* -algebra. We prove that $\mathfrak{K}_{top}^{(2)}$ is a C^* -algebra invariant. In fact we have the following result.

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Theorem 3.2: Suppose that \mathcal{A} is a unital C^* -algebra and $\{x_1, \dots, x_n\}$, $\{y_1, \dots, y_p\}$ are two families of self-adjoint generators of \mathcal{A} . Then

$$\mathfrak{K}_{top}^{(2)}(x_1, \dots, x_n) = \mathfrak{K}_{top}^{(2)}(y_1, \dots, y_p).$$

Moreover, we show in **Theorem 3.1** that if z_1, \dots, z_m is a family of self-adjoint elements in a unital C^* -algebra \mathcal{A} , then

$$\delta_{top}(z_1, \dots, z_m) \leq \max\{\mathfrak{K}_{top}^{(2)}(z_1, \dots, z_m), 1\},$$

where $\delta_{top}(z_1, \dots, z_m)$ is the Voiculescu's topological free entropy dimension of z_1, \dots, z_m in \mathcal{A} . This result, together with Theorem 3.2, provides us a possible way to compute the Voiculescu's topological free entropy dimension of an arbitrary family of self-adjoint generators of a unital C^* -algebra \mathcal{A} by studying the topological orbit dimension of a specific family of self-adjoint generators in \mathcal{A} .

We then study the topological orbit dimension in tensor products and orthogonal sums of unital C^* -algebras and obtain the following results.

Theorem 3.4: Suppose that \mathcal{A} is a unital C^* -algebra and n is a positive integer. Suppose that $\mathcal{B} = \mathcal{A} \otimes \mathcal{M}_n(\mathbb{C})$. If x_1, \dots, x_m is a family of self-adjoint generators of \mathcal{A} and y_1, \dots, y_p is a family of self-adjoint generators of \mathcal{B} , then

$$\mathfrak{K}_{top}^{(2)}(y_1, \dots, y_p) \leq \mathfrak{K}_{top}^{(2)}(x_1, \dots, x_m).$$

Theorem 3.5: Suppose that \mathcal{A} and \mathcal{B} are unital C^* -algebras with a family of self-adjoint generators x_1, \dots, x_n , and y_1, \dots, y_m respectively. Suppose $\mathcal{D} = \mathcal{A} \oplus \mathcal{B}$ is the orthogonal sum of \mathcal{A} and \mathcal{B} with a family of self-adjoint generators z_1, \dots, z_p . Then,

$$\mathfrak{K}_{top}^{(2)}(z_1, \dots, z_p) \leq \mathfrak{K}_{top}^{(2)}(x_1, \dots, x_n) + \mathfrak{K}_{top}^{(2)}(y_1, \dots, y_m).$$

Next we show that topological orbit dimension is always majorized by orbit dimension capacity:

Theorem 4.1: Suppose that \mathcal{A} is a unital C^* -algebra and x_1, \dots, x_n is a family of self-adjoint elements in \mathcal{A} . Then

$$\mathfrak{K}_{top}^{(2)}(x_1, \dots, x_n) \leq \mathfrak{K} \mathfrak{K}_2^{(2)}(x_1, \dots, x_n),$$

where $\mathfrak{K} \mathfrak{K}_2^{(2)}(x_1, \dots, x_n)$ is the orbit dimension capacity defined in Definition 4.1.

A direct consequence of Theorem 4.1 is the computation of topological free entropy dimension of every finite family of self-adjoint generators in a nuclear C^* -algebra.

Corollary 4.1: Suppose \mathcal{A} is a unital nuclear C^* -algebra with a family of self-adjoint generators x_1, \dots, x_n . If \mathcal{A} has the approximation property in the sense of Definition 4.2, then

$$\mathfrak{K}_{top}^{(2)}(x_1, \dots, x_n) = 0 \quad \text{and} \quad \delta_{top}(x_1, \dots, x_n) \leq 1.$$

The lower bound of topological free entropy dimension of a family of self-adjoint generators of a nuclear C^* -algebra depends on the choice of nuclear algebras. For example, $\delta_{top}(x_1, \dots, x_n)$

$= 0$ if x_1, \dots, x_n is a family of self-adjoint generators of the unitization of C^* -algebra of compact operators (see Theorem 5.6 in [18]). On the other hand, $\delta_{top}(x_1, \dots, x_n) = 1$ if x_1, \dots, x_n is a family of self-adjoint generators of a UHF algebra (see Theorem 5.4 in [18]).

More applications of Theorem 4.1 can be found in Corollary 4.2, Corollary 4.3 and Corollary 4.4.

The last part of the paper is devoted to prove that the topological free entropy dimension is additive in unital full free products of unital C^* -algebras with the approximation property in the sense of Definition 4.2, or equivalently in unital full free products of unital Blackadar and Kirchberg's MF algebras (See Theorem 5.1). As a corollary of Theorem 5.1, we obtain the following result, which is a generalization of an earlier result proved by Voiculescu in [37].

Corollary 5.1: Suppose that \mathcal{A}_i ($i = 1, 2, \dots, m$) is a unital C^* -algebra generated by a self-adjoint element x_i in \mathcal{A}_i . Let \mathcal{D} be the unital full free product of $\mathcal{A}_1, \dots, \mathcal{A}_n$ equipped with unital embedding from each \mathcal{A}_i into \mathcal{D} . If we identify the element x_i in \mathcal{A}_i with its image in \mathcal{D} , then

$$\delta_{top}(x_1, \dots, x_n) = \sum_{i=1}^n \delta_{top}(x_i) = n - \sum_{i=1}^n \frac{1}{n_i},$$

where n_i is the number of elements in the spectrum of x_i in \mathcal{A}_i (We use the notation $1/\infty = 0$).

The concept of MF algebras was introduced by Blackadar and Kirchberg in [1]. This class of C^* -algebras plays an important role in the classification of C^* -algebras and it is connected to Brown, Douglas and Fillmore's extension theory (see the striking result of Haagerup and Thorbjensen on $Ext(C_r^*(F_2))$). In the appendix, we show that the unital full free product of two Blackadar and Kirchberg's separable unital MF C^* -algebras is again an MF algebra (See Theorem 5.4). Based on Haagerup and Thorbjensen's work on $Ext(C_r^*(F_2))$, we are able to conclude that $Ext(C_r^*(F_2) *_{\mathbb{C}} C_r^*(F_2))$ is not a group. This result provides us new example of C^* -algebra whose extension semigroup is not a group.

The organization of the paper is as follows. In section 2, we give the definitions of topological free entropy dimension and topological orbit dimension of n -tuple of elements in a unital C^* -algebra. Some properties of topological orbit dimension are discussed in section 3. In section 4, we introduce the concept of orbit dimensional capacity and discuss its application in the computations of topological orbit dimension in finitely generated nuclear C^* algebras and several other classes of unital C^* -algebras. In section 5, we prove that topological free entropy dimension is additive in unital full free products of some unital C^* algebras. In the appendix, we show that the unital full free product of two MF algebras is again an MF algebra and $Ext(C_r^*(F_2) *_{\mathbb{C}} C_r^*(F_2))$ is not a group.

2. Definitions and preliminary

In this section, we are going to recall Voiculescu's definition of topological free entropy dimension of n -tuple of elements in a unital C^* -algebra and give the definition of topological orbit dimension of n -tuple of elements in a unital C^* -algebra.

2.1. A Covering of a set in a metric space. Suppose (X, d) is a metric space and K is a subset of X . A family of balls in X is called a covering of K if the union of these balls covers K and the centers of these balls lie in K .

2.2. Covering numbers in complex matrix algebra $(\mathcal{M}_k(\mathbb{C}))^n$. Let $\mathcal{M}_k(\mathbb{C})$ be the $k \times k$ full matrix algebra with entries in \mathbb{C} , and τ_k be the normalized trace on $\mathcal{M}_k(\mathbb{C})$, i.e., $\tau_k = \frac{1}{k}Tr$, where Tr is the usual trace on $\mathcal{M}_k(\mathbb{C})$. Let $\mathcal{U}(k)$ denote the group of all unitary matrices in $\mathcal{M}_k(\mathbb{C})$. Let $\mathcal{M}_k(\mathbb{C})^n$ denote the direct sum of n copies of $\mathcal{M}_k(\mathbb{C})$. Let $\mathcal{M}_k^{s,a}(\mathbb{C})$ be the subset of $\mathcal{M}_k(\mathbb{C})$ consisting of all self-adjoint matrices of $\mathcal{M}_k(\mathbb{C})$. Let $(\mathcal{M}_k^{s,a}(\mathbb{C}))^n$ be the direct sum (or orthogonal sum) of n copies of $\mathcal{M}_k^{s,a}(\mathbb{C})$. Let $\|\cdot\|$ be the operator norm on $\mathcal{M}_k(\mathbb{C})^n$ defined by

$$\|(A_1, \dots, A_n)\| = \max\{\|A_1\|, \dots, \|A_n\|\}$$

for all (A_1, \dots, A_n) in $\mathcal{M}_k(\mathbb{C})^n$. Let $\|\cdot\|_2$ denote the trace norm induced by τ_k on $\mathcal{M}_k(\mathbb{C})^n$, i.e.,

$$\|(A_1, \dots, A_n)\|_2 = \sqrt{\tau_k(A_1^* A_1) + \dots + \tau_k(A_n^* A_n)}$$

for all (A_1, \dots, A_n) in $\mathcal{M}_k(\mathbb{C})^n$.

For every $\omega > 0$, we define the ω - $\|\cdot\|$ -ball $Ball(B_1, \dots, B_n; \omega, \|\cdot\|)$ centered at (B_1, \dots, B_n) in $\mathcal{M}_k(\mathbb{C})^n$ to be the subset of $\mathcal{M}_k(\mathbb{C})^n$ consisting of all (A_1, \dots, A_n) in $\mathcal{M}_k(\mathbb{C})^n$ such that

$$\|(A_1, \dots, A_n) - (B_1, \dots, B_n)\| < \omega.$$

DEFINITION 2.1. Suppose that Σ is a subset of $\mathcal{M}_k(\mathbb{C})^n$. We define the covering number $\nu_\infty(\Sigma, \omega)$ to be the minimal number of ω - $\|\cdot\|$ -balls that constitute a covering of Σ in $\mathcal{M}_k(\mathbb{C})^n$.

For every $\omega > 0$, we define the ω - $\|\cdot\|_2$ -ball $Ball(B_1, \dots, B_n; \omega, \|\cdot\|_2)$ centered at (B_1, \dots, B_n) in $\mathcal{M}_k(\mathbb{C})^n$ to be the subset of $\mathcal{M}_k(\mathbb{C})^n$ consisting of all (A_1, \dots, A_n) in $\mathcal{M}_k(\mathbb{C})^n$ such that

$$\|(A_1, \dots, A_n) - (B_1, \dots, B_n)\|_2 < \omega.$$

DEFINITION 2.2. Suppose that Σ is a subset of $\mathcal{M}_k(\mathbb{C})^n$. We define the covering number $\nu_2(\Sigma, \omega)$ to be the minimal number of ω - $\|\cdot\|_2$ -balls that constitute a covering of Σ in $\mathcal{M}_k(\mathbb{C})^n$.

2.3. Unitary orbits of balls in $\mathcal{M}_k(\mathbb{C})^n$. For every $\omega > 0$, we define the ω -orbit- $\|\cdot\|_2$ -ball $\mathcal{U}(B_1, \dots, B_n; \omega, \|\cdot\|_2)$ centered at (B_1, \dots, B_n) in $\mathcal{M}_k(\mathbb{C})^n$ to be the subset of $\mathcal{M}_k(\mathbb{C})^n$ consisting of all (A_1, \dots, A_n) in $\mathcal{M}_k(\mathbb{C})^n$ such that there exists a unitary matrix W in $\mathcal{U}(k)$ satisfying

$$\|(A_1, \dots, A_n) - (WB_1W^*, \dots, WB_nW^*)\|_2 < \omega.$$

DEFINITION 2.3. Suppose that Σ is a subset of $\mathcal{M}_k(\mathbb{C})^n$. We define the covering number $o_2(\Sigma, \omega)$ to be the minimal number of ω -orbit- $\|\cdot\|_2$ -balls that constitute a covering of Σ in $\mathcal{M}_k(\mathbb{C})^n$.

2.4. Noncommutative polynomials. In this article, we always assume that \mathcal{A} is a unital C^* -algebra. Let $x_1, \dots, x_n, y_1, \dots, y_m$ be self-adjoint elements in \mathcal{A} . Let $\mathbb{C}\langle X_1, \dots, X_n, Y_1, \dots, Y_m \rangle$ be the set of all noncommutative polynomials in the indeterminates $X_1, \dots, X_n, Y_1, \dots, Y_m$. Let $\{P_r\}_{r=1}^\infty$ be the collection of all noncommutative polynomials in $\mathbb{C}\langle X_1, \dots, X_n, Y_1, \dots, Y_m \rangle$ with rational complex coefficients. (Here “rational complex coefficients” means that the real and imaginary parts of all coefficients of P_r are rational numbers).

REMARK 2.1. *We always assume that $1 \in \mathbb{C}\langle X_1, \dots, X_n, Y_1, \dots, Y_m \rangle$.*

2.5. Voiculescu’s norm-microstates Space. For all integers $r, k \geq 1$, real numbers $R, \epsilon > 0$ and noncommutative polynomials P_1, \dots, P_r , we define

$$\Gamma_R^{(top)}(x_1, \dots, x_n, y_1, \dots, y_m; k, \epsilon, P_1, \dots, P_r)$$

to be the subset of $(\mathcal{M}_k^{s,a}(\mathbb{C}))^{n+m}$ consisting of all these

$$(A_1, \dots, A_n, B_1, \dots, B_m) \in (\mathcal{M}_k^{s,a}(\mathbb{C}))^{n+m}$$

satisfying

$$\max\{\|A_1\|, \dots, \|A_n\|, \|B_1\|, \dots, \|B_m\|\} \leq R$$

and

$$|\|P_j(A_1, \dots, A_n, B_1, \dots, B_m)\| - \|P_j(x_1, \dots, x_n, y_1, \dots, y_m)\|| \leq \epsilon, \quad \forall 1 \leq j \leq r.$$

Define the norm-microstates space of x_1, \dots, x_n in the presence of y_1, \dots, y_m , denoted by

$$\Gamma_R^{(top)}(x_1, \dots, x_n : y_1, \dots, y_m; k, \epsilon, P_1, \dots, P_r),$$

to be the projection of $\Gamma_R^{(top)}(x_1, \dots, x_n, y_1, \dots, y_m; k, \epsilon, P_1, \dots, P_r)$ onto the space $(\mathcal{M}_k^{s,a}(\mathbb{C}))^n$ via the mapping

$$(A_1, \dots, A_n, B_1, \dots, B_m) \rightarrow (A_1, \dots, A_n).$$

2.6. Voiculescu’s topological free entropy dimension (see [37]). Define

$$\nu_\infty(\Gamma_R^{(top)}(x_1, \dots, x_n : y_1, \dots, y_m; k, \epsilon, P_1, \dots, P_r), \omega)$$

to be the covering number of the set $\Gamma_R^{(top)}(x_1, \dots, x_n : y_1, \dots, y_m; k, \epsilon, P_1, \dots, P_r)$ by ω - $\|\cdot\|$ -balls in the metric space $(\mathcal{M}_k^{s,a}(\mathbb{C}))^n$ equipped with operator norm.

DEFINITION 2.4. *Define*

$$\begin{aligned} \delta_{top}(x_1, \dots, x_n : y_1, \dots, y_m; \omega) \\ = \sup_{R>0} \inf_{\epsilon>0, r \in \mathbb{N}} \limsup_{k \rightarrow \infty} \frac{\log(\nu_\infty(\Gamma_R^{(top)}(x_1, \dots, x_n : y_1, \dots, y_m; k, \epsilon, P_1, \dots, P_r), \omega))}{-k^2 \log \omega} \end{aligned}$$

The topological free entropy dimension of x_1, \dots, x_n in the presence of y_1, \dots, y_m is defined by

$$\delta_{top}(x_1, \dots, x_n : y_1, \dots, y_m) = \limsup_{\omega \rightarrow 0^+} \delta_{top}(x_1, \dots, x_n : y_1, \dots, y_m; \omega)$$

REMARK 2.2. Let $R > \max\{\|x_1\|, \dots, \|x_n\|, \|y_1\|, \dots, \|y_m\|\}$ be a positive number. By definition, we know

$$\begin{aligned} & \delta_{top}(x_1, \dots, x_n : y_1, \dots, y_m) \\ &= \limsup_{\omega \rightarrow 0^+} \inf_{\epsilon > 0, r \in \mathbb{N}} \limsup_{k \rightarrow \infty} \frac{\log(\nu_\infty(\Gamma_R^{(top)}(x_1, \dots, x_n : y_1, \dots, y_m; k, \epsilon, P_1, \dots, P_r), \omega))}{-k^2 \log \omega} \end{aligned}$$

2.7. Topological orbit dimension $\mathfrak{K}_{top}^{(2)}$. Define

$$o_2(\Gamma_R^{(top)}(x_1, \dots, x_n : y_1, \dots, y_m; k, \epsilon, P_1, \dots, P_r), \omega)$$

to be the covering number of the set $\Gamma_R^{(top)}(x_1, \dots, x_n : y_1, \dots, y_m; k, \epsilon, P_1, \dots, P_r)$ by ω -orbit- $\|\cdot\|_2$ -balls in the metric space $(\mathcal{M}_k^{s,a}(\mathbb{C}))^n$ equipped with the trace norm.

DEFINITION 2.5. Define

$$\begin{aligned} & \mathfrak{K}_{top}^{(2)}(x_1, \dots, x_n : y_1, \dots, y_m; \omega) \\ &= \sup_{R > 0} \inf_{\epsilon > 0, r \in \mathbb{N}} \limsup_{k \rightarrow \infty} \frac{\log(o_2(\Gamma_R^{(top)}(x_1, \dots, x_n : y_1, \dots, y_m; k, \epsilon, P_1, \dots, P_r), \omega))}{k^2} \end{aligned}$$

REMARK 2.3. The value of $\mathfrak{K}_{top}^{(2)}(x_1, \dots, x_n : y_1, \dots, y_m; \omega)$ increases as ω decreases.

The topological orbit dimension of x_1, \dots, x_n in the presence of y_1, \dots, y_m is defined by

$$\mathfrak{K}_{top}^{(2)}(x_1, \dots, x_n : y_1, \dots, y_m) = \limsup_{\omega \rightarrow 0^+} \mathfrak{K}_{top}^{(2)}(x_1, \dots, x_n : y_1, \dots, y_m; \omega)$$

REMARK 2.4. In the notation $\mathfrak{K}_{top}^{(2)}$, the subscript “top” stands for the norm-microstates space and superscript “(2)” stands for the using of unitary-orbit- $\|\cdot\|_2$ -balls when counting the covering numbers of the norm-microstates spaces.

REMARK 2.5. Let $R > \max\{\|x_1\|, \dots, \|x_n\|, \|y_1\|, \dots, \|y_m\|\}$ be a positive number. By definition, we know

$$\begin{aligned} & \mathfrak{K}_{top}^{(2)}(x_1, \dots, x_n : y_1, \dots, y_m) \\ &= \limsup_{\omega \rightarrow 0^+} \inf_{\epsilon > 0, r \in \mathbb{N}} \limsup_{k \rightarrow \infty} \frac{\log(o_2(\Gamma_R^{(top)}(x_1, \dots, x_n : y_1, \dots, y_m; k, \epsilon, P_1, \dots, P_r), \omega))}{k^2} \end{aligned}$$

2.8. C*-algebra ultraproducts. Suppose $\{\mathcal{M}_{k_m}(\mathbb{C})\}_{m=1}^\infty$ is a sequence of complex matrix algebras where k_m goes to infinity as m goes to infinity. Let γ be a free ultrafilter in $\beta(\mathbb{N}) \setminus \mathbb{N}$. We can introduce a unital C*-algebra $\prod_{m=1}^\infty \mathcal{M}_{k_m}(\mathbb{C})$ as follows:

$$\prod_{m=1}^\infty \mathcal{M}_{k_m}(\mathbb{C}) = \{(Y_m)_{m=1}^\infty \mid \forall m \geq 1, Y_m \in \mathcal{M}_{k_m}(\mathbb{C}) \text{ and } \sup_{m \geq 1} \|Y_m\| < \infty\}.$$

We can also introduce a norm closed two sided ideal \mathcal{I}_∞ as follows.

$$\mathcal{I}_\infty = \{(Y_m)_{m=1}^\infty \in \prod_{m=1}^\infty \mathcal{M}_{k_m}(\mathbb{C}) \mid \lim_{m \rightarrow \gamma} \|Y_m\| = 0\}$$

DEFINITION 2.6. The C^* -algebra ultraproduct of $\{\mathcal{M}_{k_m}(\mathbb{C})\}_{m=1}^\infty$ along the ultrfilter γ , denoted by $\prod_{m=1}^\gamma \mathcal{M}_{k_m}(\mathbb{C})$, is defined to be the quotient algebra, $\prod_{m=1}^\infty \mathcal{M}_{k_m}(\mathbb{C})/\mathcal{I}_\infty$, of $\prod_{m=1}^\infty \mathcal{M}_{k_m}(\mathbb{C})$ by the ideal \mathcal{I}_∞ . The image of $(Y_m)_{m=1}^\infty \in \prod_{m=1}^\infty \mathcal{M}_{k_m}(\mathbb{C})$ in $\prod_{m=1}^\gamma \mathcal{M}_{k_m}(\mathbb{C})$ is denoted by $[(Y_m)_m]$.

3. Properties of topological orbit dimension $\mathfrak{K}_{top}^{(2)}$

In this section, we are going to discuss properties of the topological orbit dimension $\mathfrak{K}_{top}^{(2)}$.

The following result explains the relationship between Voiculescu's topological free entropy dimension and topological orbit dimension of n -tuple of elements in a unital C^* -algebra.

LEMMA 3.1. Suppose that \mathcal{A} is a unital C^* -algebra and x_1, \dots, x_n is a family of self-adjoint elements in \mathcal{A} . If

$$\mathfrak{K}_{top}^{(2)}(x_1, \dots, x_n) < \infty,$$

then

$$\delta_{top}(x_1, \dots, x_n) \leq 1.$$

PROOF. Let $\{P_r\}_{r=1}^\infty$ be the collection of all noncommutative polynomials in $\mathbb{C}\langle X_1, \dots, X_n \rangle$ with rational complex coefficients. For any $0 < \omega < 1/10$, $R > \max\{\|x_1\|, \dots, \|x_n\|\}$, by Remark 2.3 we know that

$$\inf_{r \in \mathbb{N}, \epsilon > 0} \limsup_{k \rightarrow \infty} \frac{\log(o_2(\Gamma_R^{(top)}(x_1, \dots, x_n, k, \epsilon, P_1, \dots, P_r), \omega))}{-k^2 \log \omega} \leq \mathfrak{K}_{top}^{(2)}(x_1, \dots, x_n) \cdot \frac{1}{-\log \omega}.$$

By Szarek's results in [31], there is a family of unitary matrices $\{U_\lambda\}_{\lambda \in \Lambda}$ in $\mathcal{U}(k)$ such that (i) $\{Ball(U_\lambda; \frac{\omega}{R}, \|\cdot\|)\}_{\lambda \in \Lambda}$ is a covering of $\mathcal{U}(k)$; and (ii) the cardinality of Λ , $|\Lambda| \leq (\frac{CR}{\omega})^{k^2}$ where C is a constant independent of k, ω . Thus from relationship between covering number (see Definition 2.2) and unitary orbit covering number (see Definition 2.3), we have

$$\begin{aligned} & \inf_{r \in \mathbb{N}, \epsilon > 0} \limsup_{k \rightarrow \infty} \frac{\log(\nu_2(\Gamma_R^{(top)}(x_1, \dots, x_n, k, \epsilon, P_1, \dots, P_r), 3\omega))}{-k^2 \log \omega} \\ & \leq \inf_{r \in \mathbb{N}, \epsilon > 0} \limsup_{k \rightarrow \infty} \frac{\log(o_2(\Gamma_R^{(top)}(x_1, \dots, x_n, k, \epsilon, P_1, \dots, P_r), \omega) \cdot (\frac{CR}{\omega})^{k^2})}{-k^2 \log \omega} \\ & \leq 1 + \frac{\log C + \log R}{-\log \omega} + \mathfrak{K}_{top}^{(2)}(x_1, \dots, x_n) \cdot \frac{1}{-\log \omega}. \end{aligned}$$

Now the result of the theorem follows directly from the definitions of the topological free entropy dimension and the topological orbit dimension, together with Remark 2.2, Remark 2.5 and the remark in Section 6 of [37] (or Proposition 5.1 in [18]).

□

A direct consequence of the preceding lemma is the following theorem.

THEOREM 3.1. *Suppose that \mathcal{A} is a unital C^* -algebra and x_1, \dots, x_n is a family of self-adjoint elements in \mathcal{A} . Then*

$$\delta_{top}(x_1, \dots, x_n) \leq \max\{\mathfrak{K}_{top}^{(2)}(x_1, \dots, x_n), 1\}.$$

In particular, if

$$\mathfrak{K}_{top}^{(2)}(x_1, \dots, x_n) = 0,$$

then

$$\delta_{top}(x_1, \dots, x_n) \leq 1.$$

The following lemma will be needed in the proof of Theorem 3.2.

LEMMA 3.2. *Let $x_1, \dots, x_n, y_1, \dots, y_p$ be self-adjoint elements in a unital C^* -algebra \mathcal{A} . If y_1, \dots, y_p are in the C^* subalgebra generated by x_1, \dots, x_n in \mathcal{A} , then, for every $\omega > 0$,*

$$\mathfrak{K}_{top}^{(2)}(x_1, \dots, x_n; 4\omega) \leq \mathfrak{K}_{top}^{(2)}(x_1, \dots, x_n : y_1, \dots, y_p; 2\omega) \leq \mathfrak{K}_{top}^{(2)}(x_1, \dots, x_n; \omega).$$

PROOF. It is a straightforward adaptation of the proof of Prop. 1.6 in [35]. Suppose that $\{P_r\}_{r=1}^\infty$, and $\{Q_s\}_{s=1}^\infty$ respectively, are families of noncommutative polynomials in $\mathbb{C}\langle X_1, \dots, X_n, Y_1, \dots, Y_p \rangle$, and $\mathbb{C}\langle X_1, \dots, X_n \rangle$ respectively, with rational coefficients.

Given $R > \max_{1 \leq j \leq n} \|x_j\| + \max_{1 \leq j \leq p} \|y_j\|$, $r, s \in \mathbb{N}$ and $\epsilon > 0$, we can find $r_1, s_1 \in \mathbb{N}$ and $\epsilon_1 > 0, \epsilon_2 > 0$ such that, for all $k \in \mathbb{N}$,

$$\begin{aligned} \Gamma_R^{(top)}(x_1, \dots, x_n; k, \epsilon_1, Q_1, \dots, Q_{s_1}) &\subseteq \Gamma_R^{(top)}(x_1, \dots, x_n : y_1, \dots, y_p; k, \epsilon, P_1, \dots, P_r) \\ \Gamma_R^{(top)}(x_1, \dots, x_n : y_1, \dots, y_p; k, \epsilon_2, P_1, \dots, P_{r_1}) &\subseteq \Gamma_R^{(top)}(x_1, \dots, x_n; k, \epsilon, Q_1, \dots, Q_s). \end{aligned}$$

Hence

$$\begin{aligned} o_2(\Gamma_R^{(top)}(x_1, \dots, x_n; k, \epsilon_1, Q_1, \dots, Q_{s_1}), 4\omega) &\leq o_2(\Gamma_R^{(top)}(x_1, \dots, x_n : y_1, \dots, y_p; k, \epsilon, P_1, \dots, P_r), 2\omega) \\ o_2(\Gamma_R^{(top)}(x_1, \dots, x_n : y_1, \dots, y_p; k, \epsilon_2, P_1, \dots, P_{r_1}), 2\omega) &\leq o_2(\Gamma_R^{(top)}(x_1, \dots, x_n; k, \epsilon, Q_1, \dots, Q_s), \omega), \end{aligned}$$

for all $\omega > 0$. Therefore, for all $\omega > 0$,

$$\begin{aligned} \inf_{\epsilon_1 > 0, s_1 \in \mathbb{N}} \limsup_{k \rightarrow \infty} \frac{\log(o_2(\Gamma_R^{(top)}(x_1, \dots, x_n; k, \epsilon_1, Q_1, \dots, Q_{s_1}), 4\omega))}{k^2} \\ \leq \limsup_{k \rightarrow \infty} \frac{\log(o_2(\Gamma_R^{(top)}(x_1, \dots, x_n : y_1, \dots, y_p; k, \epsilon, P_1, \dots, P_r), 2\omega))}{k^2}; \end{aligned}$$

and

$$\begin{aligned} \inf_{\epsilon_2 > 0, r_1 \in \mathbb{N}} \limsup_{k \rightarrow \infty} \frac{\log(o_2(\Gamma_R^{(top)}(x_1, \dots, x_n : y_1, \dots, y_p; k, \epsilon_2, P_1, \dots, P_{r_1}), 2\omega))}{k^2} \\ \leq \limsup_{k \rightarrow \infty} \frac{\log(o_2(\Gamma_R^{(top)}(x_1, \dots, x_n; k, \epsilon, Q_1, \dots, Q_s), \omega))}{k^2}. \end{aligned}$$

It follows that, for all $\omega > 0$,

$$\begin{aligned} \inf_{\epsilon_1 > 0, s_1 \in \mathbb{N}} \limsup_{k \rightarrow \infty} \frac{\log(o_2(\Gamma_R^{(top)}(x_1, \dots, x_n; k, \epsilon_1, Q_1, \dots, Q_{s_1}), 4\omega))}{k^2} \\ \leq \inf_{\epsilon > 0, r \in \mathbb{N}} \limsup_{k \rightarrow \infty} \frac{\log(o_2(\Gamma_R^{(top)}(x_1, \dots, x_n; y_1, \dots, y_p; k, \epsilon, P_1, \dots, P_r), 2\omega))}{k^2}; \end{aligned}$$

and

$$\begin{aligned} \inf_{\epsilon_2 > 0, r_1 \in \mathbb{N}} \limsup_{k \rightarrow \infty} \frac{\log(o_2(\Gamma_R^{(top)}(x_1, \dots, x_n; y_1, \dots, y_p; k, \epsilon_2, P_1, \dots, P_{r_1}), 2\omega))}{k^2} \\ \leq \inf_{\epsilon > 0, s \in \mathbb{N}} \limsup_{k \rightarrow \infty} \frac{\log(o_2(\Gamma_R^{(top)}(x_1, \dots, x_n; k, \epsilon, Q_1, \dots, Q_s), \omega))}{k^2}. \end{aligned}$$

The rest follows from the definitions. \square

3.1. Our next result shows that the topological orbit dimension $\mathfrak{K}_{top}^{(2)}$ is in fact a C^* -algebra invariant.

THEOREM 3.2. *Suppose that \mathcal{A} is a unital C^* -algebra and $\{x_1, \dots, x_n\}, \{y_1, \dots, y_p\}$ are two families of self-adjoint generators of \mathcal{A} . Then*

$$\mathfrak{K}_{top}^{(2)}(x_1, \dots, x_n) = \mathfrak{K}_{top}^{(2)}(y_1, \dots, y_p).$$

PROOF. Note x_1, \dots, x_n are elements in \mathcal{A} that generate \mathcal{A} as a C^* -algebra. For every $0 < \omega < 1$, there exists a family of noncommutative polynomials $\psi_i(x_1, \dots, x_n)$, $1 \leq i \leq p$, such that

$$\left(\sum_{i=1}^p \|y_i - \psi_i(x_1, \dots, x_n)\|^2 \right)^{1/2} < \frac{\omega}{4}.$$

For such a family of polynomials ψ_1, \dots, ψ_p , and every $R > \max\{\|x_1\|, \dots, \|x_n\|, \|y_1\|, \dots, \|y_p\|\}$ there always exists a constant $D \geq 1$, depending only on R, ψ_1, \dots, ψ_p , such that

$$\left(\sum_{i=1}^p \|\psi_i(A_1, \dots, A_n) - \psi_i(B_1, \dots, B_n)\|_2^2 \right)^{1/2} \leq D \|(A_1, \dots, A_n) - (B_1, \dots, B_n)\|_2,$$

for all $(A_1, \dots, A_n), (B_1, \dots, B_n)$ in $\mathcal{M}_k(\mathbb{C})^n$, all $k \in \mathbb{N}$, satisfying $\|A_j\|, \|B_j\| \leq R$, for $1 \leq j \leq n$.

Suppose that $\{P_r\}_{r=1}^\infty$, and $\{Q_s\}_{s=1}^\infty$ respectively, is the family of noncommutative polynomials in $\mathbb{C}\langle Y_1, \dots, Y_p, X_1, \dots, X_n \rangle$, and $\mathbb{C}\langle X_1, \dots, X_n \rangle$ respectively, with rational coefficients.

For any $s \geq 1$, $\epsilon > 0$, when r is sufficiently large, ϵ' sufficiently small, every

$$(H_1, \dots, H_p, A_1, \dots, A_n)$$

in

$$\Gamma_R^{(top)}(y_1, \dots, y_p, x_1, \dots, x_n; k, \epsilon', P_1, \dots, P_r)$$

satisfies

$$\left(\sum_{i=1}^p \|H_i - \psi_i(A_1, \dots, A_n)\|^2 \right)^{1/2} \leq \frac{\omega}{4};$$

and

$$(A_1, \dots, A_n) \in \Gamma_R^{(top)}(x_1, \dots, x_n; k, \epsilon, Q_1, \dots, Q_s).$$

On the other hand, by the definition of the orbit covering number, we know there exists a set $\{\mathcal{U}(B_1^\lambda, \dots, B_n^\lambda; \frac{\omega}{4D}, \|\cdot\|_2)\}_{\lambda \in \Lambda_k}$ of $\frac{\omega}{4D}$ -orbit- $\|\cdot\|_2$ -balls that cover $\Gamma_R^{(top)}(x_1, \dots, x_n; k, \epsilon, Q_1, \dots, Q_s)$ with the cardinality of Λ_k satisfying $|\Lambda_k| = o_2(\Gamma_R^{(top)}(x_1, \dots, x_n; k, \epsilon, Q_1, \dots, Q_s), \frac{\omega}{4D})$. Thus for such (A_1, \dots, A_n) in $\Gamma_R^{(top)}(x_1, \dots, x_n; k, \epsilon, Q_1, \dots, Q_s)$, there exists some $\lambda \in \Lambda_k$ and $W \in \mathcal{U}(k)$ such that

$$\|(A_1, \dots, A_n) - (WB_1^\lambda W^*, \dots, WB_n^\lambda W^*)\|_2 \leq \frac{\omega}{4D}.$$

It follows that

$$\begin{aligned} \left(\sum_{i=1}^p \|H_i - W\psi_i(B_1^\lambda, \dots, B_n^\lambda)W^*\|_2^2 \right)^{1/2} &= \left(\sum_{i=1}^p \|H_i - \psi_i(WB_1^\lambda W^*, \dots, WB_n^\lambda W^*)\|_2^2 \right)^{1/2} \\ &\leq \left(\sum_{i=1}^p \|H_i - \psi_i(A_1, \dots, A_n)\|_2^2 \right)^{1/2} + \left(\sum_{i=1}^p \|\psi_i(A_1, \dots, A_n) - \psi_i(WB_1^\lambda W^*, \dots, WB_n^\lambda W^*)\|_2^2 \right)^{1/2} \\ &\leq \left(\sum_{i=1}^p \|H_i - \psi_i(A_1, \dots, A_n)\|^2 \right)^{1/2} + \frac{\omega}{4} \\ &\leq \frac{\omega}{2}, \end{aligned}$$

for some $\lambda \in \Lambda_k$ and $W \in \mathcal{U}(k)$, i.e.,

$$(H_1, \dots, H_p) \in \mathcal{U}(\psi_1(B_1^\lambda, \dots, B_n^\lambda), \dots, \psi_p(B_1^\lambda, \dots, B_n^\lambda); \frac{\omega}{2}).$$

Hence, for given $s \in \mathbb{N}$ and $\epsilon > 0$, when ϵ' is small enough and r is large enough,

$$\begin{aligned} o_2(\Gamma_R^{(top)}(y_1, \dots, y_p : x_1, \dots, x_n; k, \epsilon', P_1, \dots, P_r), \omega) \\ \leq |\Lambda_k| = o_2(\Gamma_R^{(top)}(x_1, \dots, x_n; k, \epsilon, Q_1, \dots, Q_s), \frac{\omega}{4D}). \end{aligned}$$

It follows that

$$\begin{aligned} \inf_{r \in \mathbb{N}, \epsilon' > 0} \lim_{k \rightarrow \infty} \frac{\log(o_2(\Gamma_R^{(top)}(y_1, \dots, y_p : x_1, \dots, x_n; k, \epsilon', P_1, \dots, P_r), \omega))}{k^2} \\ \leq \lim_{k \rightarrow \infty} \frac{\log(o_2(\Gamma_R^{(top)}(x_1, \dots, x_n; k, \epsilon, Q_1, \dots, Q_s), \frac{\omega}{4D}))}{k^2}. \end{aligned}$$

Therefore by the definition of the topological orbit dimension and Remark 2.5, we get

$$\begin{aligned}
\mathfrak{K}_{top}^{(2)}(y_1, \dots, y_p : x_1, \dots, x_n; \omega) &= \mathfrak{K}_{top}^{(2)}(y_1, \dots, y_p : x_1, \dots, x_n; \omega, R) \\
&= \inf_{\epsilon' > 0, r \in \mathbb{N}} \lim_{k \rightarrow \infty} \frac{\log(o_2(\Gamma_R^{(top)}(y_1, \dots, y_p : x_1, \dots, x_n; k, \epsilon', P_1, \dots, P_r), \omega))}{k^2} \\
&\leq \inf_{\epsilon > 0, s \in \mathbb{N}} \limsup_{k \rightarrow \infty} \frac{\log(o_2(\Gamma_R^{(top)}(x_1, \dots, x_n; k, \epsilon, Q_1, \dots, Q_s), \frac{\omega}{4D}))}{k^2} \\
&\leq \mathfrak{K}_{top}^{(2)}(x_1, \dots, x_n),
\end{aligned}$$

where the last inequality follows from the fact that $\mathfrak{K}_{top}^{(2)}(x_1, \dots, x_n; \omega)$ increases as ω decreases. Thus, by Lemma 3.2, we get

$$\mathfrak{K}_{top}^{(2)}(y_1, \dots, y_p) \leq \mathfrak{K}_{top}^{(2)}(x_1, \dots, x_n).$$

Similarly, we have

$$\mathfrak{K}_{top}^{(2)}(x_1, \dots, x_n) \leq \mathfrak{K}_{top}^{(2)}(y_1, \dots, y_p),$$

which completes the proof. \square

A slight modification of the proof of Theorem 3.2 will show the following result.

THEOREM 3.3. *Suppose that \mathcal{A} is a unital C^* -algebra with a family of self-adjoint generators y_1, \dots, y_n . Suppose \mathcal{A}_i , $j = 1, 2, \dots$ is an increasing sequence of unital C^* -subalgebras of \mathcal{A} such that $\cup_{j=1}^{\infty} \mathcal{A}_j$ is norm dense in \mathcal{A} . Suppose $x_1^{(j)}, \dots, x_{n_j}^{(j)}$ is a family of self-adjoint generators of \mathcal{A}_j for $j = 1, 2, \dots$. Then*

$$\mathfrak{K}_{top}^{(2)}(y_1, \dots, y_p) \leq \liminf_{j \rightarrow \infty} \mathfrak{K}_{top}^{(2)}(x_1^{(j)}, \dots, x_{n_j}^{(j)}).$$

PROOF. Note $\cup_{j=1}^{\infty} \mathcal{A}_j$ is norm dense in \mathcal{A} and, for each $j \geq 1$, $x_1^{(j)}, \dots, x_{n_j}^{(j)}$ is a family of self-adjoint generators of \mathcal{A}_j . For every $0 < \omega < 1$, there exist a positive integer j and a family of noncommutative polynomials $\psi_i(x_1^{(j)}, \dots, x_{n_j}^{(j)})$, $1 \leq i \leq p$, such that

$$\left(\sum_{i=1}^p \|y_i - \psi_i(x_1^{(j)}, \dots, x_{n_j}^{(j)})\|^2 \right)^{1/2} < \frac{\omega}{4}.$$

The rest of the proof is identical to the one of Theorem 3.2. \square

3.2. Suppose \mathcal{A} is a finitely generated unital C^* -algebra and n is a positive integer. In this subsection, we are going to compute the topological orbit dimension in the unital C^* -algebra $\mathcal{A} \otimes \mathcal{M}_n(\mathbb{C})$.

Assume that $\{e_{st}\}_{s,t=1}^n$ is a canonical system of matrix units in $\mathcal{M}_n(\mathbb{C})$ and I_n is the identity matrix of $\mathcal{M}_n(\mathbb{C})$.

The following statement is an easy adaption of Lemma 2.3 in [2].

LEMMA 3.3 (Lemma 2.3 in [2]). *For any $\epsilon > 0$, there is a constant $\delta > 0$ so that the following holds: For any $k \in \mathbb{N}$, if $\{E_{st}\}_{s,t=1}^n$ is a family of elements in $\mathcal{M}_k(\mathbb{C})$ satisfying:*

$$\|E_{s_1 t_1} E_{s_2 t_2} - \delta_{t_1 s_2} E_{s_1 t_2}\| \leq \delta, \quad \|E_{s_1 t_1} - E_{t_1 s_1}^*\| \leq \delta, \quad \left\| \sum_{i=1}^n E_{ii} - I_k \right\| \leq \delta, \quad \forall 1 \leq s_1, s_2, t_1, t_2 \leq n$$

(where $\delta_{t_1 s_2}$ is 1 if $t_1 = s_2$ and is 0 if $t_1 \neq s_2$), then (i) $n|k$; and (ii) there is some unitary matrix W in $\mathcal{M}_k(\mathbb{C})$ such that

$$\sum_{s,t=1}^n \|W^* E_{st} W - I_{k/n} \otimes e_{st}\| \leq \epsilon.$$

LEMMA 3.4. *Suppose \mathcal{A} is a unital C^* -algebra generated by a family of self-adjoint elements x_1, \dots, x_m . Suppose that $\{P_r\}_{r=1}^\infty$ and $\{Q_s\}_{s=1}^\infty$ respectively, is the family of noncommutative polynomials in $\mathbb{C}\langle X_1, \dots, X_m, \{Y_{st}\}_{s,t=1}^n \rangle$, and $\mathbb{C}\langle X_1, \dots, X_m \rangle$ respectively, with rational coefficients.*

Let $R > \max\{\|x_1\|, \dots, \|x_m\|, 1\}$. For any $\omega > 0$, $r_0 > 0$ and $\epsilon_0 > 0$, there are some $r > 0$ and $\epsilon > 0$ such that the following holds: if

$$(A_1, \dots, A_m, \{E_{st}\}_{s,t=1}^n) \in \Gamma_R^{(top)}(x_1 \otimes I_n, \dots, x_m \otimes I_n, \{I_A \otimes e_{st}\}_{s,t=1}^n, k, \epsilon, P_1, \dots, P_r) \neq \emptyset,$$

then (i) $n|k$; (ii) there are a unitary matrix W in $\mathcal{M}_k(\mathbb{C})$ and

$$(B_1, \dots, B_m) \in \Gamma_R^{(top)}(x_1, \dots, x_m, \frac{k}{n}, \epsilon_0, Q_1, \dots, Q_{r_0}).$$

such that

$$\sum_{s,t=1}^n \|W^* E_{st} W - I_{k/n} \otimes e_{st}\| + \sum_{i=1}^m \|W^* A_i W - B_i \otimes I_n\| \leq \omega.$$

PROOF. We will prove the result by using contradiction. Suppose, to the contrary, that the result of the lemma does not hold. There are $\omega > 0$, $r_0 \in \mathbb{N}$, and $\epsilon_0 > 0$ such that, for any $r \in \mathbb{N}$, there are $k_r \in \mathbb{N}$ and

$$(A_1^{(r)}, \dots, A_m^{(r)}, \{E_{st}^{(r)}\}_{s,t=1}^n) \in \Gamma_R^{(top)}(x_1 \otimes I_n, \dots, x_m \otimes I_n, \{I_A \otimes e_{st}\}_{s,t=1}^n, k_r, 1/r, P_1, \dots, P_r) \neq \emptyset,$$

satisfying either $n \nmid k_r$, or if W is a unitary matrix in $\mathcal{M}_{k_r}(\mathbb{C})$ and

$$(B_1, \dots, B_m) \in \Gamma_R^{(top)}(x_1, \dots, x_m, \frac{k}{n}, \epsilon_0, Q_1, \dots, Q_{r_0}). \quad (3.1)$$

then

$$\sum_{s,t=1}^n \|W^* E_{st}^{(r)} W - I_{k/n} \otimes e_{st}\| + \sum_{i=1}^m \|W^* A_i^{(r)} W - B_i \otimes I_n\| > \omega. \quad (3.2)$$

Let γ be a free ultra-filter in $\beta(\mathbb{N}) \setminus \mathbb{N}$. Let $\prod_{r=1}^\gamma \mathcal{M}_{k_r}(\mathbb{C})$ be the C^* algebra ultra-product of matrices algebras $(\mathcal{M}_{k_r}(\mathbb{C}))_{r=1}^\infty$ along the ultra-filter γ , i.e. $\prod_{r=1}^\gamma \mathcal{M}_{k_r}(\mathbb{C})$ is the quotient algebra of the C^* -algebra $\prod_r \mathcal{M}_{k_r}(\mathbb{C})$ by \mathcal{I}_∞ , where $\mathcal{I}_\infty = \{(Y_r)_{r=1}^\infty \in \prod_r \mathcal{M}_{k_r}(\mathbb{C}) \mid \lim_{r \rightarrow \gamma} \|Y_r\| = 0\}$.

Let ψ be the $*$ -isomorphism from the C^* -algebra $\mathcal{A} \otimes \mathcal{M}_n(\mathbb{C})$ into the C^* -algebra $\prod_{r=1}^{\gamma} \mathcal{M}_{k_r}(\mathbb{C})$ induced by the mapping

$$x_i \otimes I_n \rightarrow [(A_i^{(r)})_r] \in \prod_{r=1}^{\gamma} \mathcal{M}_{k_r}(\mathbb{C}), \quad I_{\mathcal{A}} \otimes e_{st} \rightarrow [(E_{st}^{(r)})_r] \in \prod_{r=1}^{\gamma} \mathcal{M}_{k_r}(\mathbb{C}) \quad \forall 1 \leq i \leq m, 1 \leq s, t \leq n.$$

Thus $\{\psi(I_{\mathcal{A}} \otimes e_{st})\}_{s,t=1}^n$ is also a system of matrix units of a C^* -subalgebra ($*$ -isomorphic to $\mathcal{M}_n(\mathbb{C})$) in $\prod_{r=1}^{\gamma} \mathcal{M}_{k_r}(\mathbb{C})$. By the preceding lemma, without loss of generality, we can assume that $n|k_r$ and there is a sequence of unitary matrices $\{W_r\}_{r=1}^{\infty}$ where W_r is in $\mathcal{M}_{k_r}(\mathbb{C})$ such that

$$[(E_{st}^{(r)})_r] = [(W_r(I_{k_r/n} \otimes e_{st})W_r^*)_r], \quad \forall 1 \leq s, t \leq n. \quad (3.3)$$

Note that

$$[(A_i^{(r)})_r][(E_{st}^{(r)})_r] = [(E_{st}^{(r)})_r][(A_i^{(r)})_r], \quad \forall 1 \leq i \leq m, 1 \leq s, t \leq n.$$

Thus by (3.3), there are $B_1^{(r)}, \dots, B_m^{(r)}$ in $\mathcal{M}_{k_r/n}(\mathbb{C})$ for each $r \geq 1$ such that

$$[(A_i^{(r)})_r] = [(W_r(B_i^{(r)} \otimes I_n)W_r^*)_r], \quad \forall 1 \leq i \leq m,$$

which contradicts with our assumptions (3.1), (3.2) and (3.3). This completes the proof of the lemma. \square

Now we are ready to prove the main result in this subsection.

THEOREM 3.4. *Suppose that \mathcal{A} is a unital C^* -algebra and n is a positive integer. Suppose that $\mathcal{B} = \mathcal{A} \otimes \mathcal{M}_n(\mathbb{C})$. If x_1, \dots, x_m is a family of self-adjoint generators of \mathcal{A} and y_1, \dots, y_p is a family of self-adjoint generators of \mathcal{B} , then*

$$\mathfrak{K}_{top}^{(2)}(y_1, \dots, y_p) \leq \mathfrak{K}_{top}^{(2)}(x_1, \dots, x_m).$$

PROOF. Suppose that $\{P_r\}_{r=1}^{\infty}$, and $\{Q_s\}_{s=1}^{\infty}$ respectively, is a family of noncommutative polynomials in $\mathbb{C}\langle X_1, \dots, X_m, \{Y_{st}\}_{s,t=1}^n \rangle$, and $\mathbb{C}\langle X_1, \dots, X_m \rangle$ respectively, with rational coefficients.

Let $R > \max\{\|x_1\|, \dots, \|x_m\|, 1\}$. For any $\omega > 0$, $r_0 > 0$ and $\epsilon_0 > 0$, by the preceding lemma, there is a $r > 0$ such that, $\forall k \in \mathbb{N}$,

$$\begin{aligned} o_2(\Gamma_R^{(top)}(x_1 \otimes I_n, \dots, x_m \otimes I_n, \{I_{\mathcal{A}} \otimes e_{st}\}_{s,t=1}^n; k, 1/r, P_1, \dots, P_r, 2\omega)) \\ \leq o_2(\Gamma_R^{(top)}(x_1, \dots, x_n; k, 1/r_0, Q_1, \dots, Q_{r_0}, \omega)). \end{aligned}$$

Thus,

$$\begin{aligned} \inf_{r \in \mathbb{N}} \limsup_{k \rightarrow \infty} \frac{\log(o_2(\Gamma_R^{(top)}(x_1 \otimes I_n, \dots, x_m \otimes I_n, \{I_{\mathcal{A}} \otimes e_{st}\}_{s,t=1}^n; k, 1/r, P_1, \dots, P_r, 2\omega))}{k^2} \\ \leq \limsup_{k \rightarrow \infty} \frac{\log(o_2(\Gamma_R^{(top)}(x_1, \dots, x_n; k, 1/r_0, Q_1, \dots, Q_{r_0}, \omega))}{k^2}. \end{aligned}$$

So

$$\begin{aligned} & \inf_{r \in \mathbb{N}} \limsup_{k \rightarrow \infty} \frac{\log(o_2(\Gamma_R^{(top)}(x_1 \otimes I_n, \dots, x_m \otimes I_n, \{I_{\mathcal{A}} \otimes e_{st}\}_{s,t=1}^n; k, 1/r, P_1, \dots, P_r, 2\omega))}{k^2} \\ & \leq \inf_{r_0 \in \mathbb{N}} \limsup_{k \rightarrow \infty} \frac{\log(o_2(\Gamma_R^{(top)}(x_1, \dots, x_n; k, 1/r_0, Q_1, \dots, Q_{r_0}, \omega))}{k^2}. \end{aligned}$$

It follows easily that

$$\mathfrak{K}_{top}^{(2)}(x_1 \otimes I_n, \dots, x_m \otimes I_n, \{I_{\mathcal{A}} \otimes e_{st}\}_{s,t=1}^n) \leq \mathfrak{K}_{top}^{(2)}(x_1, \dots, x_m).$$

By Theorem 3.2, we have

$$\mathfrak{K}_{top}^{(2)}(y_1, \dots, y_p) \leq \mathfrak{K}_{top}^{(2)}(x_1, \dots, x_m),$$

where y_1, \dots, y_p is a family of self-adjoint generators of \mathcal{B} □

The following corollary follows directly from the preceding theorem.

COROLLARY 3.1. *Suppose that \mathcal{A} is a unital C^* -algebra with a family of self-adjoint generators x_1, \dots, x_m . Suppose that n is a positive integer and $\mathcal{B} = \mathcal{A} \otimes \mathcal{M}_n(\mathbb{C})$. If*

$$\mathfrak{K}_{top}^{(2)}(x_1, \dots, x_m) = 0,$$

then

$$\mathfrak{K}_{top}^{(2)}(y_1, \dots, y_p) = 0 \quad \text{and} \quad \delta_{top}(y_1, \dots, y_p) \leq 1,$$

where y_1, \dots, y_p is any family of self-adjoint generators of \mathcal{B} .

EXAMPLE 3.1. *Suppose that x_1, \dots, x_m is a family of self-adjoint generators of a full matrix algebra $\mathcal{M}_n(\mathbb{C})$. Then*

$$\mathfrak{K}_{top}^{(2)}(x_1, \dots, x_m) = 0.$$

3.3. In this subsection, we assume that \mathcal{A} and \mathcal{B} are two unital C^* -algebras and $\mathcal{A} \oplus \mathcal{B}$ is the orthogonal sum of \mathcal{A} and \mathcal{B} . We assume x_1, \dots, x_n , or y_1, \dots, y_m , is a family of self-adjoint generators of \mathcal{A} , or \mathcal{B} respectively. Suppose that $\{P_r\}_{r=1}^\infty$, and $\{Q_s\}_{s=1}^\infty$ respectively, is the family of noncommutative polynomials in $\mathbb{C}\langle X_1, \dots, X_n \rangle$, and $\mathbb{C}\langle Y_1, \dots, Y_m \rangle$ respectively, with rational coefficients. Suppose that $\{S_r\}_{r=1}^\infty$ is the family of noncommutative polynomials in $\mathbb{C}\langle X_1, \dots, X_n, Y_1, \dots, Y_m \rangle$ with rational coefficients.

Let $R > \max\{\|x_1\|, \dots, \|x_n\|, \|y_1\|, \dots, \|y_m\|\}$ be a positive number. By the definition of topological orbit dimension, we have the following.

LEMMA 3.5. *Let*

$$\alpha > \mathfrak{K}_{top}^{(2)}(x_1, \dots, x_n) \quad \text{and} \quad \beta > \mathfrak{K}_{top}^{(2)}(y_1, \dots, y_m).$$

(i) For each $\omega > 0$, there is $r(\omega)$ satisfying

$$\limsup_{k_1 \rightarrow \infty} \frac{\log(o_2(\Gamma_R^{(top)}(x_1, \dots, x_n; k_1, \frac{1}{r(\omega)}, P_1, \dots, P_{r(\omega)}), \omega))}{k_1^2} < \alpha;$$

$$\limsup_{k_2 \rightarrow \infty} \frac{\log(o_2(\Gamma_R^{(top)}(y_1, \dots, y_m; k_2, \frac{1}{r(\omega)}, Q_1, \dots, Q_{r(\omega)}), \omega))}{k_2^2} < \beta.$$

(ii) Therefore, for each $\omega > 0$ and $r(\omega) \in \mathbb{N}$, there is some $K(r(\omega)) \in \mathbb{N}$ satisfying

$$\log(o_2(\Gamma_R^{(top)}(x_1, \dots, x_n; k_1, \frac{1}{r(\omega)}, P_1, \dots, P_{r(\omega)}), \omega)) < \alpha k_1^2, \quad \forall k_1 \geq K(r(\omega));$$

$$\log(o_2(\Gamma_R^{(top)}(y_1, \dots, y_m; k_2, \frac{1}{r(\omega)}, Q_1, \dots, Q_{r(\omega)}), \omega)) < \beta k_2^2, \quad \forall k_2 \geq K(r(\omega)).$$

LEMMA 3.6. Suppose that \mathcal{A} and \mathcal{B} are two unital C^* algebras and x_1, \dots, x_n , or y_1, \dots, y_m is a family of self-adjoint elements that generates \mathcal{A} , or \mathcal{B} respectively.

Let $R > \max\{\|x_1\|, \dots, \|x_n\|, \|y_1\|, \dots, \|y_m\|\}$ be a positive number. For any $\omega > 0$, $r_0 \in \mathbb{N}$, there is some $t > 0$ so that the following holds: $\forall r > t, \forall k \geq 1$, if

$$(X_1, \dots, X_n, Y_1, \dots, Y_m) \in \Gamma_R^{(top)}(x_1 \oplus 0, \dots, x_n \oplus 0, 0 \oplus y_1, \dots, 0 \oplus y_m; k, \frac{1}{r}, S_1, \dots, S_r),$$

then there are

$$(A_1, \dots, A_n) \in \Gamma_R^{(top)}(x_1, \dots, x_n; k_1, \frac{1}{r_0}, P_1, \dots, P_{r_0}),$$

$$(B_1, \dots, B_m) \in \Gamma_R^{(top)}(y_1, \dots, y_m; k_2, \frac{1}{r_0}, Q_1, \dots, Q_{r_0})$$

and $U \in \mathcal{U}(k)$ so that (i) $k_1 + k_2 = k$; and (ii)

$$\|(X_1, \dots, X_n, Y_1, \dots, Y_m) - U^*(A_1 \oplus 0, \dots, A_n \oplus 0, 0 \oplus B_1, \dots, 0 \oplus B_m)U\| < \omega.$$

PROOF. The proof of this lemma is a slight modification of the one of Lemma 4.2 in [18]. \square

THEOREM 3.5. Suppose that \mathcal{A} and \mathcal{B} are unital C^* -algebras with a family of self-adjoint generators x_1, \dots, x_n , or y_1, \dots, y_m respectively. Suppose $\mathcal{D} = \mathcal{A} \oplus \mathcal{B}$ is the orthogonal sum of \mathcal{A} and \mathcal{B} with a family of self-adjoint generators z_1, \dots, z_p . Then,

$$\mathfrak{K}_{top}^{(2)}(z_1, \dots, z_p) \leq \mathfrak{K}_{top}^{(2)}(x_1, \dots, x_n) + \mathfrak{K}_{top}^{(2)}(y_1, \dots, y_m).$$

PROOF. Recall that $\{S_r\}_{r=1}^\infty$, $\{P_r\}_{r=1}^\infty$, and $\{Q_s\}_{s=1}^\infty$ respectively, are the families of noncommutative polynomials in $\mathbb{C}\langle X_1, \dots, X_n, Y_1, \dots, Y_m \rangle$, $\mathbb{C}\langle X_1, \dots, X_n \rangle$, and $\mathbb{C}\langle Y_1, \dots, Y_m \rangle$ respectively, with rational coefficients.

Let

$$\alpha > \mathfrak{K}_{top}^{(2)}(x_1, \dots, x_n) \quad \text{and} \quad \beta > \mathfrak{K}_{top}^{(2)}(y_1, \dots, y_m),$$

and $R > \max\{\|x_1\|, \dots, \|x_n\|, \|y_1\|, \dots, \|y_m\|\}$ be a positive number. By definition, the values of topological orbit dimension $\mathfrak{K}_{top}^{(2)}$ can only be $-\infty$ or ≥ 0 . Without loss of generality, we

can assume that $\alpha > 0$ and $\beta > 0$. By Lemma 3.5, for any $\omega > 0$, there are $r(\omega) \in \mathbb{N}$ and $K(r(\omega)) \in \mathbb{N}$ satisfying

$$o_2(\Gamma_R^{(top)}(x_1, \dots, x_n; k_1, \frac{1}{r(\omega)}, P_1, \dots, P_{r(\omega)}), \omega) < e^{\alpha k_1^2}, \quad \forall k_1 \geq K(r(\omega)); \quad (3.4)$$

$$o_2(\Gamma_R^{(top)}(y_1, \dots, y_m; k_2, \frac{1}{r(\omega)}, Q_1, \dots, Q_{r(\omega)}), \omega) < e^{\beta k_2^2}, \quad \forall k_2 \geq K(r(\omega)). \quad (3.5)$$

On the other hand, for each $\omega > 0$ and $r(\omega) \in \mathbb{N}$, it follows from Lemma 3.6 that there is some $t \in \mathbb{N}$ so that $\forall r > t, \forall k \geq 1$, if

$$(X_1, \dots, X_n, Y_1, \dots, Y_m) \in \Gamma_R^{(top)}(x_1 \oplus 0, \dots, x_n \oplus 0, 0 \oplus y_1, \dots, 0 \oplus y_m; k, \frac{1}{r}, S_1, \dots, S_r),$$

then there are

$$(A_1, \dots, A_n) \in \Gamma_R^{(top)}(x_1, \dots, x_n; k_1, \frac{1}{r_\omega}, P_1, \dots, P_{r_\omega}),$$

$$(B_1, \dots, B_m) \in \Gamma_R^{(top)}(y_1, \dots, y_m; k_2, \frac{1}{r_\omega}, Q_1, \dots, Q_{r_\omega})$$

and $U \in \mathcal{U}(k)$ so that (i) $k_1 + k_2 = k$; and (ii)

$$\|(X_1, \dots, X_n, Y_1, \dots, Y_m) - U^*(A_1 \oplus 0, \dots, A_n \oplus 0, 0 \oplus B_1, \dots, 0 \oplus B_m)U\| < \omega.$$

It follows that

$$\begin{aligned} & o_2(\Gamma_R^{(top)}(x_1 \oplus 0, \dots, x_n \oplus 0, 0 \oplus y_1, \dots, 0 \oplus y_m; k, \frac{1}{r}, S_1, \dots, S_r), 3\omega) \\ & \leq \sum_{k_1+k_2=k} \left(o_2(\Gamma_R^{(top)}(x_1, \dots, x_n; k_1, \frac{1}{r_\omega}, P_1, \dots, P_{r_\omega}), \omega) \right. \\ & \quad \cdot o_2(\Gamma_R^{(top)}(y_1, \dots, y_m; k_2, \frac{1}{r_\omega}, Q_1, \dots, Q_{r_\omega}), \omega) \Big) \\ & = \left(\sum_{k_1=1}^{K(r(\omega))} + \sum_{k_1=K(r(\omega))+1}^{k-K(r(\omega))-1} + \sum_{k_1=k-K(r(\omega))}^k \right) \left(o_2(\Gamma_R^{(top)}(x_1, \dots, x_n; k_1, \frac{1}{r_\omega}, P_1, \dots, P_{r_\omega}), \omega) \right. \\ & \quad \cdot o_2(\Gamma_R^{(top)}(y_1, \dots, y_m; k_2, \frac{1}{r_\omega}, Q_1, \dots, Q_{r_\omega}), \omega) \Big) \quad (3.6) \end{aligned}$$

Let

$$M_\omega = \max_{1 \leq k_1 \leq K(r(\omega))} o_2(\Gamma_R^{(top)}(x_1, \dots, x_n; k_1, \frac{1}{r_\omega}, P_1, \dots, P_{r_\omega}), \omega) + 1,$$

$$N_\omega = \max_{1 \leq k_2 \leq K(r(\omega))} o_2(\Gamma_R^{(top)}(y_1, \dots, y_m; k_2, \frac{1}{r_\omega}, Q_1, \dots, Q_{r_\omega}), \omega) + 1$$

By (3.4) and (3.5), we know that

$$\begin{aligned}
(3.6) &\leq K(r(\omega))M_\omega e^{\beta k_2^2} + K(r(\omega))N_\omega e^{\alpha k_1^2} + (k - 2K(r(\omega))) \cdot (e^{\alpha k_1^2 + \beta k_2^2} + 1) \\
&\leq K(r(\omega))M_\omega e^{\beta k^2} + K(r(\omega))N_\omega e^{\alpha k^2} + 2k \cdot e^{(\alpha+\beta)k^2} \\
&\leq 3k \cdot e^{(\alpha+\beta)k^2},
\end{aligned}$$

when k is large enough. Now it is not hard to show that

$$\mathfrak{K}_{top}^{(2)}(x_1 \oplus 0, \dots, x_n \oplus 0, 0 \oplus y_1, \dots, 0 \oplus y_m) \leq \alpha + \beta.$$

Thus, by Theorem 3.2, we have

$$\mathfrak{K}_{top}^{(2)}(z_1, \dots, z_p) \leq \mathfrak{K}_{top}^{(2)}(x_1, \dots, x_m) + \mathfrak{K}_{top}^{(2)}(y_1, \dots, y_m),$$

where z_1, \dots, z_p is any family of self-adjoint generators of $\mathcal{A} \oplus \mathcal{B}$. \square

The following corollary follows directly from the preceding theorem.

COROLLARY 3.2. *Suppose that \mathcal{A} and \mathcal{B} are unital C^* -algebras with a family of self-adjoint generators x_1, \dots, x_n , and y_1, \dots, y_m respectively. Suppose $\mathcal{D} = \mathcal{A} \oplus \mathcal{B}$ is the orthogonal sum of \mathcal{A} and \mathcal{B} with a family of self-adjoint generators z_1, \dots, z_p . if*

$$\mathfrak{K}_{top}^{(2)}(x_1, \dots, x_m) = \mathfrak{K}_{top}^{(2)}(y_1, \dots, y_m) = 0,$$

then

$$\mathfrak{K}_{top}^{(2)}(z_1, \dots, z_p) = 0 \quad \text{and} \quad \delta_{top}(z_1, \dots, z_p) \leq 1.$$

By Example 3.1 and Corollary 3.2, we have the following result.

EXAMPLE 3.2. *Suppose x_1, \dots, x_n is a family of self-adjoint generators of a finite dimensional C^* -algebra \mathcal{B} . Then*

$$\mathfrak{K}_{top}^{(2)}(x_1, \dots, x_n) = 0.$$

REMARK 3.1. *The result in the preceding example will be extended to the general case of a nuclear C^* -algebra in Corollary 4.1.*

4. Orbit dimension capacity

In this section, we are going to define the concept of “orbit dimension capacity” of n -tuple of elements in a unital C^* -algebra, which is an analogue of “free dimension capacity” in [37].

4.1. Modified free orbit dimension in finite von Neumann algebras. Let \mathcal{M} be a von Neumann algebra with a tracial state τ , and x_1, \dots, x_n be self-adjoint elements in \mathcal{M} . For any positive R and ϵ , and any m, k in \mathbb{N} , let $\Gamma_R(x_1, \dots, x_n; m, k, \epsilon; \tau)$ be the subset of $\mathcal{M}_k^{s,a}(\mathbb{C})^n$ consisting of all (A_1, \dots, A_n) in $\mathcal{M}_k^{s,a}(\mathbb{C})^n$ such that $\max_{1 \leq j \leq n} \|A_j\| \leq R$, and

$$|\tau_k(A_{i_1} \cdots A_{i_q}) - \tau(x_{i_1} \cdots x_{i_q})| < \epsilon,$$

for all $1 \leq i_1, \dots, i_q \leq n$, and $1 \leq q \leq m$.

For any $\omega > 0$, let $o_2(\Gamma_R(x_1, \dots, x_n; m, k, \epsilon; \tau), \omega)$ be the minimal number of ω -orbit- $\|\cdot\|_2$ -balls in $\mathcal{M}_k(\mathbb{C})^n$ that constitute a covering of $\Gamma_R(x_1, \dots, x_n; m, k, \epsilon; \tau)$.

Now we define, successively,

$$\begin{aligned}\mathfrak{K}_2^{(2)}(x_1, \dots, x_n; \omega; \tau) &= \sup_{R>0} \inf_{m \in \mathbb{N}, \epsilon > 0} \limsup_{k \rightarrow \infty} \frac{\log(o_2(\Gamma_R(x_1, \dots, x_n; m, k, \epsilon; \tau), \omega))}{k^2} \\ \mathfrak{K}_2^{(2)}(x_1, \dots, x_n; \tau) &= \limsup_{\omega \rightarrow 0^+} \mathfrak{K}_2^{(2)}(x_1, \dots, x_n; \omega; \tau),\end{aligned}$$

where $\mathfrak{K}_2^{(2)}(x_1, \dots, x_n; \tau)$ is called the *modified free orbit-dimension* of x_1, \dots, x_n with respect to the tracial state τ .

REMARK 4.1. *If the von Neumann algebra \mathcal{M} with a tracial state τ is replaced by a unital C^* -algebra \mathcal{A} with a tracial state τ , then $\mathfrak{K}_2^{(2)}(x_1, \dots, x_n; \tau)$ is still well-defined.*

From the previous definition, it follows directly our next result.

LEMMA 4.1. *Suppose x_1, \dots, x_n is a family of self-adjoint elements in a von Neumann algebra with a tracial state τ . Let $\mathfrak{K}_2(x_1, \dots, x_n; \tau)$ be the upper orbit dimension of x_1, \dots, x_n defined in Definition 1 of [17]. We have, if*

$$\mathfrak{K}_2(x_1, \dots, x_n; \tau) = 0,$$

then

$$\mathfrak{K}_2^{(2)}(x_1, \dots, x_n; \tau) = 0.$$

4.2. Definition of orbit dimension capacity. We are ready to give the definition of “orbit dimension capacity”.

DEFINITION 4.1. *Suppose that \mathcal{A} is a unital C^* -algebra and $TS(\mathcal{A})$ is the set of all tracial states of \mathcal{A} . Suppose that x_1, \dots, x_n is a family of self-adjoint elements in \mathcal{A} . Define*

$$\mathfrak{K}\mathfrak{K}_2^{(2)}(x_1, \dots, x_n) = \sup_{\tau \in TS(\mathcal{A})} \mathfrak{K}_2^{(2)}(x_1, \dots, x_n; \tau)$$

to be the orbit dimension capacity of x_1, \dots, x_n .

4.3. Topological orbit dimension is majorized by orbit dimension capacity. We have the following relationship between topological orbit dimension and orbit dimension capacity.

THEOREM 4.1. *Suppose that \mathcal{A} is a unital C^* -algebra and x_1, \dots, x_n is a family of self-adjoint elements in \mathcal{A} . Then*

$$\mathfrak{K}_{top}^{(2)}(x_1, \dots, x_n) \leq \mathfrak{K}\mathfrak{K}_2^{(2)}(x_1, \dots, x_n).$$

PROOF. The proof is a slight modification of the one in section 3 of [37]. For the sake of the completeness, we also include Voiculescu’s arguments here.

If $\mathfrak{K}_{top}^{(2)}(x_1, \dots, x_n) = -\infty$, there is nothing to prove. We might assume that

$$\mathfrak{K}_{top}^{(2)}(x_1, \dots, x_n) > \alpha > -\infty.$$

We will show that

$$\mathfrak{K}_2^{(2)}(x_1, \dots, x_n) = \sup_{\tau \in TS(\mathcal{A})} \mathfrak{K}_2^{(2)}(x_1, \dots, x_n; \tau) > \alpha.$$

Let $\{P_r\}_{r=1}^\infty$ be a family of noncommutative polynomials in $\mathbb{C}\langle X_1, \dots, X_n \rangle$ with rational coefficients. Let $R > \max\{\|x_1\|, \dots, \|x_n\|\}$. From the assumption that $\mathfrak{K}_{top}^{(2)}(x_1, \dots, x_n) > \alpha$, it follows that there exist a positive number $\omega_0 > 0$ and a sequence of positive integers $\{k_q\}_{q=1}^\infty$ with $k_1 < k_2 < \dots$, so that for some $\alpha' > \alpha$,

$$\lim_{q \rightarrow \infty} \frac{\log(o_2(\Gamma_R^{(top)}(x_1, \dots, x_n; k_q, \frac{1}{q}, P_1, \dots, P_q), \omega_0))}{k_q^2} > \alpha'.$$

Let $\mathcal{A}(n)$ be the universal unital C^* -algebra generated by self-adjoint elements a_1, \dots, a_n of norm R , that is the unital full free product of n copies of $C[-R, R]$. A microstate

$$\eta = (A_1, \dots, A_n) \in \Gamma_R^{(top)}(x_1, \dots, x_n; k_q, \frac{1}{q}, P_1, \dots, P_q) = \Gamma(q)$$

defines a unital $*$ -homomorphism $\psi_\eta : \mathcal{A}(n) \rightarrow \mathcal{M}_{k_q}(\mathbb{C})$ so that $\psi_\eta(a_i) = A_i$ ($1 \leq i \leq n$) and a tracial state $\tau_\eta \in TS(\mathcal{A}(n))$ with

$$\tau_\eta = \frac{Tr_{k_q} \circ \psi_\eta}{k_q}.$$

Similarly there is a $*$ -homomorphism $\psi : \mathcal{A}(n) \rightarrow \mathcal{A}$ so that $\psi(a_i) = x_i$, for $1 \leq i \leq n$.

It is not hard to see that the weak topology on $\Omega = TS(\mathcal{A}(n))$ is induced by the metric

$$d(\tau_1, \tau_2) = \sum_{s=1}^{\infty} \sum_{(i_1, \dots, i_s) \in (\{1, \dots, n\})^s} (2Rn)^{-s} |(\tau_1 - \tau_2)(a_{i_1} \cdots a_{i_s})|.$$

Therefore, Ω is a compact metric space and

$$K_q = \{\tau_\eta \in \Omega \mid \eta \in \Gamma(q)\}$$

is a compact subset of Ω because $\eta \rightarrow \tau_\eta$ is continuous and $\Gamma(q)$ is compact. Let further $K \subseteq \Omega$ denote the compact subset $(TS(\mathcal{A})) \circ \psi$.

Given $\epsilon > 0$, from the fact that Ω is compact it follows that there is some $L(\epsilon) > 0$ so that for each $q \geq 1$,

$$K_q = K_q^1 \cup K_q^2 \cup \dots \cup K_q^{L(\epsilon)}$$

where each compact set K_q^j has diameter $< \epsilon$. Let

$$\Gamma(q, j) = \{\eta \in \Gamma(q) \mid \tau_\eta \in K_q^j\}.$$

We have

$$\Gamma(q) = \Gamma(q, 1) \cup \dots \cup \Gamma(q, L(\epsilon)).$$

Let further $\Gamma'(q)$ denote some $\Gamma(q, j)$ such that

$$o_2(\Gamma'(q), \omega_0) \geq \frac{o_2(\Gamma(q), \omega_0)}{L(\epsilon)}.$$

Thus we have

$$\lim_{q \rightarrow \infty} \frac{\log o_2(\Gamma'(q), \omega_0)}{k_q^2} > \alpha'.$$

Given ϵ successively the values $1, 1/2, 1/3, \dots, 1/s, \dots$, we can find a subsequence $\{q_s\}_{s=1}^\infty$ such that the chosen set $K_{q_s}^{j_s} \subseteq K_{q_s}$ has diameter $< \frac{1}{s}$ and the corresponding set $\Gamma'(q_s)$ satisfying

$$\lim_{s \rightarrow \infty} \frac{\log o_2(\Gamma'(q_s), \omega_0)}{k_{q_s}^2} > \alpha'.$$

Without loss of generality, we can assume that τ is the weak limit of some sequence $(\tau_{\eta(q_s)})_{s=1}^\infty$. Then $\tau \in K$. In fact,

$$\begin{aligned} |\tau(Q(a_1, \dots, a_n))| &= \lim_{s \rightarrow \infty} |\tau_{\eta(q_s)}(Q(a_1, \dots, a_n))| \\ &\leq \limsup_{s \rightarrow \infty} \|\psi_{\eta(q_s)}(Q(a_1, \dots, a_n))\| \\ &\leq \lim_{s \rightarrow \infty} \left(\frac{1}{s} + \|Q(x_1, \dots, x_n)\| \right) \\ &= \|Q(x_1, \dots, x_n)\| \\ &= \|\psi(Q(a_1, \dots, a_n))\|. \end{aligned}$$

Now it follows from the density of the polynomials Q in $\mathcal{A}(n)$ that $\tau \in K$.

We can further assume that there is a subsequence $\{q_{s(t)}\}_{t=1}^\infty$ of $\{q_s\}_{s=1}^\infty$ so that the chosen set $K_{q_{s(t)}}^{j_{s(t)}} \subseteq K_{q_{s(t)}}$ is $\subseteq B(\tau, 1/t)$, the ball of radius $1/t$ and center τ . Therefore, for any $m \in \mathbb{N}$ and $\epsilon > 0$, we have

$$\Gamma'(q_{s(t)}) \subseteq \Gamma_R(x_1, \dots, x_n; k_{q_{s(t)}}, m, \epsilon; \tau)$$

when t is large enough. Thus

$$\mathfrak{K}_2^{(2)}(x_1, \dots, x_n; \tau) \geq \mathfrak{K}_2^{(2)}(x_1, \dots, x_n; \omega_0; \tau) \geq \lim_{t \rightarrow \infty} \frac{\log o_2(\Gamma'(q_{s(t)}), \omega_0)}{k_{q_{s(t)}}^2} > \alpha'$$

and hence

$$\mathfrak{K}\mathfrak{K}_2^{(2)}(x_1, \dots, x_n) = \sup_{\tau \in TS(\mathcal{A})} \mathfrak{K}_2^{(2)}(x_1, \dots, x_n; \tau) \geq \mathfrak{K}_{top}^{(2)}(x_1, \dots, x_n).$$

□

4.4. Nuclear C^* -algebras. Recall the definition of approximation property of a unital C^* -algebra in [18] as follows.

DEFINITION 4.2. Suppose \mathcal{A} is a unital C^* algebra and x_1, \dots, x_n is a family of self-adjoint elements of \mathcal{A} that generates \mathcal{A} as a C^* -algebra. Let $\{P_r\}_{r=1}^\infty$ be a family of noncommutative polynomials in $\mathbb{C}\langle X_1, \dots, X_n \rangle$ with rational coefficients. If for any $R > \max\{\|x_1\|, \dots, \|x_n\|\}$, $r > 0$, $\epsilon > 0$, there is a sequence of positive integers $k_1 < k_2 < \dots$ such that

$$\Gamma_R^{(top)}(x_1, \dots, x_n; k_s, \epsilon, P_1, \dots, P_r) \neq \emptyset, \quad \forall s \geq 1$$

then \mathcal{A} is called having approximation property.

Now we can compute the topological free entropy dimension of a family of self-adjoint generators in a unital nuclear C^* -algebra.

COROLLARY 4.1. *Suppose \mathcal{A} is a unital nuclear C^* -algebra with a family of self-adjoint generators x_1, \dots, x_n . If \mathcal{A} has the approximation property in the sense of Definition 4.2, then*

$$\mathfrak{K}_{top}^{(2)}(x_1, \dots, x_n) = 0 \quad \text{and} \quad \delta_{top}(x_1, \dots, x_n) \leq 1.$$

PROOF. It is known that every representation of a nuclear C^* -algebra yields an injective von Neumann algebra. From Lemma 4.1 and Theorem 2 in [17], it follows that

$$\mathfrak{K}_2^{(2)}(x_1, \dots, x_n; \tau) = 0, \quad \forall \tau \in TS(\mathcal{A}),$$

where $TS(\mathcal{A})$ is the set of all tracial states of \mathcal{A} . Then, by Theorem 4.1 we know that

$$\mathfrak{K}_{top}^{(2)}(x_1, \dots, x_n) = 0.$$

By Theorem 3.1,

$$\delta_{top}(x_1, \dots, x_n) \leq 1.$$

□

REMARK 4.2. By [1] (see also Theorem 5.2 in Appendix), we know that a nuclear C^* -algebra \mathcal{A} has the approximation property in the sense of Definition 4.2 if and only if \mathcal{A} is an NF algebra with a finite family of generators. Thus Corollary 4.1 can be restated as the following: If \mathcal{A} is an NF algebra with a family of self-adjoint generators x_1, \dots, x_n . Then

$$\mathfrak{K}_{top}^{(2)}(x_1, \dots, x_n) = 0 \quad \text{and} \quad \delta_{top}(x_1, \dots, x_n) \leq 1.$$

4.5. Tensor products. In this subsection, we are going to prove the following result.

COROLLARY 4.2. *Suppose that \mathcal{B}_1 and \mathcal{B}_2 are two unital C^* -algebras and $\mathcal{A}_1, \mathcal{A}_2$ with $1 \in \mathcal{A}_1 \subseteq \mathcal{B}_1, 1 \in \mathcal{A}_2 \subseteq \mathcal{B}_2$ are infinite dimensional, unital, simple C^* -subalgebras with a unique tracial state. Suppose that $\mathcal{B} = \mathcal{B}_1 \otimes_\nu \mathcal{B}_2$ is the C^* -tensor product of \mathcal{B}_1 and \mathcal{B}_2 with respect to a cross norm $\|\cdot\|_\nu$. If \mathcal{B} has the approximation property in the sense of Definition 4.2, then*

$$\mathfrak{K}_{top}^{(2)}(x_1, \dots, x_n) = 0 \quad \text{and} \quad \delta_{top}(x_1, \dots, x_n) = 1,$$

where x_1, \dots, x_n is any family of self-adjoint generators of \mathcal{B} .

PROOF. Assume that τ is a tracial state of \mathcal{B} and \mathcal{H} is the Hilbert $L^2(\mathcal{B}, \tau)$. Let ψ be the GNS representation of \mathcal{B} on \mathcal{H} . Note that $\mathcal{A}_1, \mathcal{A}_2$ are infinite dimensional, unital simple C^* -algebras with a unique tracial state. It is not hard to see that both $\psi(\mathcal{A}_1)$ and $\psi(\mathcal{A}_2)$ generate diffuse finite von Neumann algebras on \mathcal{H} . Thus both $\psi(\mathcal{B}_1)$ and $\psi(\mathcal{B}_2)$ generate diffuse finite von Neumann algebras on \mathcal{H} . Moreover, $\psi(\mathcal{B}_1)$ and $\psi(\mathcal{B}_2)$ commute with each other. Thus, by Corollary 4 in [17], we have that

$$\mathfrak{K}_2(\psi(x_1), \dots, \psi(x_n); \tau) = 0,$$

where $\mathfrak{K}_2(\psi(x_1), \dots, \psi(x_n); \tau)$ is the upper free orbit dimension of $\psi(x_1), \dots, \psi(x_n)$ (with respect to τ) defined in [17]. By Lemma 3.1, we have

$$\mathfrak{K}_2^{(2)}(\psi(x_1), \dots, \psi(x_n); \tau) = 0.$$

Since τ is an arbitrary tracial state of \mathcal{B} , we have

$$\mathfrak{K}\mathfrak{K}_2^{(2)}(x_1, \dots, x_n) = 0.$$

By Theorem 3.1, we have

$$\mathfrak{K}_{top}^{(2)}(x_1, \dots, x_n) = 0.$$

Therefore,

$$\delta_{top}(x_1, \dots, x_n) \leq 1.$$

On the other hand, a consequence of Theorem 5.2 in [18] says that

$$\delta_{top}(x_1, \dots, x_n) \geq 1,$$

if \mathcal{B} has approximation property in the sense of Definition 4.2. Hence

$$\mathfrak{K}_{top}^{(2)}(x_1, \dots, x_n) = 0 \quad \text{and} \quad \delta_{top}(x_1, \dots, x_n) = 1,$$

where x_1, \dots, x_n is any family of self-adjoint generators of \mathcal{B} . □

EXAMPLE 4.1. Assume that \mathcal{A} is a finite generated unital C^* -algebra with the approximation property in the sense of Definition 4.2. By Theorem 5.2, Theorem 5.4 in Appendix and Proposition 3.1 in [20], we know that $(C_r^*(F_2) *_{\mathbb{C}} \mathcal{A}) \otimes_{min} C_r^*(F_2)$ has the approximation property, where $C_r^*(F_2)$ is the reduced C^* -algebra of free group F_2 . Thus, by Corollary 4.2, we know that, for any family of self-adjoint generators x_1, \dots, x_n of $(C_r^*(F_2) *_{\mathbb{C}} \mathcal{A}) \otimes_{min} C_r^*(F_2)$,

$$\delta_{top}(x_1, \dots, x_n) = 1.$$

Similar argument as the preceding corollary shows the following result.

COROLLARY 4.3. Suppose that \mathcal{B}_1 is a unital C^* -algebra and \mathcal{B}_2 is an infinite dimension, unital simple C^* -algebra with a unique tracial state τ . Suppose that $\mathcal{B} = \mathcal{B}_1 \otimes_{\nu} \mathcal{B}_2$ is the C^* -tensor product of \mathcal{B}_1 and \mathcal{B}_2 with respect to a cross norm $\|\cdot\|_{\nu}$. Suppose that z_1, \dots, z_p is a family of self-adjoint generators of \mathcal{B}_2 and x_1, \dots, x_n is a family of self-adjoint generators of \mathcal{B} . If \mathcal{B} has the approximation property in the sense of Definition 4.2 and $\mathfrak{K}_2^{(2)}(z_1, \dots, z_p; \tau) = 0$, then

$$\mathfrak{K}_{top}^{(2)}(x_1, \dots, x_n) = 0 \quad \text{and} \quad \delta_{top}(x_1, \dots, x_n) = 1.$$

EXAMPLE 4.2. Assume that \mathcal{A} is a UHF algebra, or an irrational rotation algebra, or $C_r^*(F_2) \otimes_{min} C_r^*(F_2)$ and \mathcal{B} is a finitely generated unital C^* -algebra with the approximation property in the sense of Definition 4.2. Then, by [1] or [20], $\mathcal{A} \otimes_{min} \mathcal{B}$ has the approximation property. Suppose that x_1, \dots, x_n is a family of self-adjoint generators of $\mathcal{A} \otimes_{min} \mathcal{B}$. Then

$$\delta_{top}(x_1, \dots, x_n) = 1.$$

EXAMPLE 4.3. By Theorem 3.1 of [20], $C^*(F_2) \otimes_{\max} (C_r^*(F_2) \otimes_{\min} C_r^*(F_2))$ has the approximation property, where $C^*(F_2)$ is the full C^* -algebra of the free group F_2 . Suppose that x_1, \dots, x_n is a family of self-adjoint generators of $C^*(F_2) \otimes_{\max} (C_r^*(F_2) \otimes_{\min} C_r^*(F_2))$. Then

$$\delta_{\text{top}}(x_1, \dots, x_n) = 1.$$

4.6. Crossed products. In this subsection, we are going to prove the following result.

COROLLARY 4.4. Suppose that \mathcal{A} is an infinite dimensional unital simple C^* -algebra with a unique tracial state τ . Suppose G is a countable group of actions $\{\alpha_g\}_{g \in G}$ on \mathcal{A} . Suppose that $\mathcal{D} = \mathcal{A} \rtimes G$ is either full or reduced crossed product of \mathcal{A} by the actions of G . Suppose that z_1, \dots, z_p is a family of self-adjoint generators of \mathcal{A} and x_1, \dots, x_n is a family of self-adjoint generators of \mathcal{D} . If \mathcal{D} has the approximation property in the sense of Definition 4.2 and $\mathfrak{K}_2^{(2)}(z_1, \dots, z_p; \tau) = 0$, then

$$\mathfrak{K}_{\text{top}}^{(2)}(x_1, \dots, x_n) = 0 \quad \text{and} \quad \delta_{\text{top}}(x_1, \dots, x_n) = 1.$$

PROOF. Assume that τ_1 is a tracial state of \mathcal{D} and \mathcal{H} is the Hilbert $L^2(\mathcal{D}, \tau_1)$. Let ψ be the GNS representation of \mathcal{D} on \mathcal{H} . Note that \mathcal{A} an infinite dimensional unital simple C^* -algebra with a unique tracial state τ . Thus $\tau_1|_{\mathcal{A}} = \tau$. It is not hard to see that $\psi(\mathcal{A})$ generates a diffuse finite von Neumann algebra on \mathcal{H} . Moreover, for any $g \in G$,

$$\psi(g^{-1})\psi(\mathcal{A})\psi(g) \subseteq \psi(\mathcal{A}).$$

It follows from the fact that $\mathfrak{K}_2^{(2)}(z_1, \dots, z_p; \tau) = 0$ and Theorem 4 in [17], that

$$\mathfrak{K}_2^{(2)}(\psi(x_1), \dots, \psi(x_n); \tau_1) = 0.$$

Since τ_1 is an arbitrary tracial state of \mathcal{D} , we have

$$\mathfrak{K}\mathfrak{K}_2^{(2)}(x_1, \dots, x_n) = 0.$$

By Theorem 3.1, we have

$$\mathfrak{K}_{\text{top}}^{(2)}(x_1, \dots, x_n) = 0.$$

Therefore,

$$\delta_{\text{top}}(x_1, \dots, x_n) \leq 1.$$

On the other hand, a consequence of Theorem 5.2 in [18] says that

$$\delta_{\text{top}}(x_1, \dots, x_n) \geq 1,$$

if \mathcal{D} has approximation property in the sense of Definition 4.2. Hence

$$\mathfrak{K}_{\text{top}}^{(2)}(x_1, \dots, x_n) = 0 \quad \text{and} \quad \delta_{\text{top}}(x_1, \dots, x_n) = 1,$$

where x_1, \dots, x_n is any family of self-adjoint generators of \mathcal{D} . □

EXAMPLE 4.4. Let $C_r^*(F_2) \otimes_{\min} C_r^*(F_2)$ be the reduced C^* -algebra of the group $F_2 \times F_2$. Let u_1, u_2 , or v_1, v_2 respectively, be the canonical unitary generators of the left copy, or the right copy respectively, of $C_r^*(F_2)$ and $0 < \theta < 1$ be a positive number. Let α be a homomorphism from \mathbb{Z} into $\text{Aut}(C_r^*(F_2) \otimes_{\min} C_r^*(F_2))$ induced by the following mapping: $\forall n \in \mathbb{Z}, j = 1, 2$

$$\alpha(n)(u_j) = e^{2n\pi\theta \cdot i} u_j \quad \text{and} \quad \alpha(n)(v_j) = e^{2n\pi\theta \cdot i} v_j.$$

Then, by Theorem 4.2 of [20], $(C_r^*(F_2) \otimes_{\min} C_r^*(F_2)) \rtimes_{\alpha} \mathbb{Z}$ has the approximation property. Therefore, by Corollary 4.4, we have

$$\delta_{\text{top}}(x_1, \dots, x_n) = 1,$$

where x_1, \dots, x_n is any family of self-adjoint generators of $(C_r^*(F_2) \otimes_{\min} C_r^*(F_2)) \rtimes_{\alpha} \mathbb{Z}$.

5. Topological free entropy dimension in full free products of unital C^* -algebras

Assume that $\{\mathcal{A}_i\}_{i=1}^m$ ($m \geq 2$) is a family of unital C^* -algebras. Recall the definition of unital full free product of $\mathcal{A}_1, \dots, \mathcal{A}_m$ as follows.

DEFINITION 5.1. The unital full free product of the unital C^* -algebras $\{\mathcal{A}_i\}_{i=1}^m$ ($m \geq 2$) is a unital C^* -algebra \mathcal{D} equipped with unital embedding $\{\sigma_i : \mathcal{A}_i \rightarrow \mathcal{D}\}_{i=1}^m$, such that (i) the set $\cup_{i=1}^m \sigma_i(\mathcal{A}_i)$ is norm dense in \mathcal{D} ; and (ii) if ϕ_i is a unital $*$ -homomorphism from \mathcal{A}_i into a unital C^* -algebra $\bar{\mathcal{D}}$ for $i = 1, 2, \dots, m$, then there is a unital $*$ -homomorphism ψ from \mathcal{D} to $\bar{\mathcal{D}}$ satisfying $\phi_i = \psi \circ \sigma_i$, for $i = 1, 2, \dots, m$.

In this section, for a family of positive integer n_1, \dots, n_m , we will let $\{X_j^{(i)}\}_{1 \leq i \leq m; 1 \leq j \leq n_i}$ be a family of indeterminates and $\{P_r\}_{r=1}^{\infty}$ be a family of noncommutative polynomials in $\mathbb{C}\langle X_j^{(i)} : 1 \leq i \leq m; 1 \leq j \leq n_i \rangle$ with rational coefficients. For each $1 \leq i \leq m$ and $j \geq 1$, let $P_j^{(i)}$ be a polynomial in $X_1^{(i)}, \dots, X_{n_i}^{(i)}$ defined by:

$$P_j^{(i)}(X_1^{(i)}, \dots, X_{n_i}^{(i)}) = P_j(0, \dots, 0, X_1^{(i)}, \dots, X_{n_i}^{(i)}, 0, \dots, 0).$$

LEMMA 5.1. Suppose $\{\mathcal{A}_i\}_{i=1}^m$ ($m \geq 2$) is a family of unital C^* -algebras and \mathcal{D} is the unital full free product of the C^* -algebras $\{\mathcal{A}_i\}_{i=1}^m$ equipped with the unital embedding $\{\sigma_i : \mathcal{A}_i \rightarrow \mathcal{D}\}_{i=1}^m$. Suppose $\{x_j^{(i)}\}_{j=1}^{n_i}$ is a family of self-adjoint generators of \mathcal{A}_i for $1 \leq i \leq m$. Then

$$\delta_{\text{top}}(\sigma_1(x_1^{(1)}), \dots, \sigma_1(x_{n_1}^{(1)}), \dots, \sigma_m(x_1^{(m)}), \dots, \sigma_m(x_{n_m}^{(m)})) \leq \sum_{i=1}^m \delta_{\text{top}}(x_1^{(i)}, \dots, x_{n_i}^{(i)}).$$

PROOF. Let $R > \max_{1 \leq i \leq m, 1 \leq j \leq n_i} \|x_j^{(i)}\|$ be a positive number. For any $r \in \mathbb{N}$ and $\epsilon > 0$, there is a positive integer r_1 such that

$$\begin{aligned} & \Gamma_R^{(\text{top})}(\sigma_1(x_1^{(1)}), \dots, \sigma_1(x_{n_1}^{(1)}), \dots, \sigma_m(x_1^{(m)}), \dots, \sigma_m(x_{n_m}^{(m)}); k, \epsilon, P_1, \dots, P_{r_1}) \\ & \subseteq \Gamma_R^{(\text{top})}(\sigma_1(x_1^{(1)}), \dots, \sigma_1(x_{n_1}^{(1)}); k, \epsilon, P_1^{(1)}, \dots, P_{r_1}^{(1)}) \oplus \\ & \quad \dots \oplus \Gamma_R^{(\text{top})}(\sigma_m(x_1^{(m)}), \dots, \sigma_m(x_{n_m}^{(m)}); k, \epsilon, P_1^{(m)}, \dots, P_{r_1}^{(m)}) \\ & = \Gamma_R^{(\text{top})}(x_1^{(1)}, \dots, x_{n_1}^{(1)}; k, \epsilon, P_1^{(1)}, \dots, P_{r_1}^{(1)}) \oplus \dots \oplus \Gamma_R^{(\text{top})}(x_1^{(m)}, \dots, x_{n_m}^{(m)}; k, \epsilon, P_1^{(m)}, \dots, P_{r_1}^{(m)}), \end{aligned}$$

where

$$P_j^{(i)}(\sigma_i(x_1^{(i)}), \dots, \sigma_i(x_{n_i}^{(i)})) = P_j(0, \dots, 0, \sigma_i(x_1^{(i)}), \dots, \sigma_i(x_{n_i}^{(i)}), 0, \dots, 0), \quad 1 \leq j \leq r, 1 \leq i \leq m.$$

By the definition of topological free entropy dimension, we get that

$$\delta_{top}(\sigma_1(x_1^{(1)}), \dots, \sigma_1(x_{n_1}^{(1)}), \dots, \sigma_m(x_1^{(m)}), \dots, \sigma_m(x_{n_m}^{(m)})) \leq \sum_{i=1}^m \delta_{top}(x_1^{(i)}, \dots, x_{n_i}^{(i)}).$$

□

5.1. Suppose that \mathcal{A} is a unital C^* -algebra and x_1, \dots, x_n is a family of self-adjoint elements in \mathcal{A} . Recall Voiculescu's semi-microstates as follows. Suppose that $\{Q_r\}_{r=1}^\infty$ is a family of noncommutative polynomials in $\mathbb{C}\langle X_1, \dots, X_n \rangle$ with rational coefficients. Let $R, \epsilon > 0$, $r, k \in \mathbb{N}$. Define

$$\Gamma_R^{(top1/2)}(x_1, \dots, x_n; k, \epsilon, Q_1, \dots, Q_r)$$

to be the subset of $(\mathcal{M}_k^{s,a}(\mathbb{C}))^n$ consisting of all these

$$(A_1, \dots, A_n) \in (\mathcal{M}_k^{s,a}(\mathbb{C}))^n$$

satisfying

$$\max\{\|A_1\|, \dots, \|A_n\|\} \leq R$$

and

$$\|Q_j(A_1, \dots, A_n)\| \leq \|Q_j(x_1, \dots, x_n)\| + \epsilon, \quad \forall 1 \leq j \leq r.$$

It is easy to see that

$$\Gamma_R^{(top)}(x_1, \dots, x_n; k, \epsilon, Q_1, \dots, Q_r) \subseteq \Gamma_R^{(top1/2)}(x_1, \dots, x_n; k, \epsilon, Q_1, \dots, Q_r).$$

LEMMA 5.2. Suppose $\{\mathcal{A}_i\}_{i=1}^m$ ($m \geq 2$) is a family of unital C^* -algebras and \mathcal{D} is the full free product of the unital C^* -algebras $\{\mathcal{A}_i\}_{i=1}^m$ equipped with the unital embedding $\{\sigma_i : \mathcal{A}_i \rightarrow \mathcal{D}\}_{i=1}^m$. Suppose $\{x_j^{(i)}\}_{j=1}^{n_i}$ is a family of self-adjoint generators of \mathcal{A}_i for $1 \leq i \leq m$. Let $R > \max\{\|x_j^{(i)}\|, 1 \leq i \leq m, 1 \leq j \leq n_i\}$ be a positive number. For any $r_0 \in \mathbb{N}$ and $\epsilon_0 > 0$, there are $r_1 \in \mathbb{N}$ and $\epsilon_1 > 0$ such that, for any $k \in \mathbb{N}$, if

$$(A_1^{(i)}, \dots, A_{n_i}^{(i)}) \in \Gamma_R^{(top1/2)}(x_1^{(i)}, \dots, x_{n_i}^{(i)}; k, \epsilon_1, P_1^{(i)}, \dots, P_{r_1}^{(i)}), \quad \text{for } 1 \leq i \leq m,$$

where $P_1^{(i)}, \dots, P_{r_1}^{(i)}$ is defined as in Lemma 5.1, then

$$(A_1^{(1)}, \dots, A_{n_1}^{(1)}, \dots, A_1^{(m)}, \dots, A_{n_m}^{(m)}) \\ \in \Gamma_R^{(top1/2)}(\sigma_1(x_1^{(1)}), \dots, \sigma_1(x_{n_1}^{(1)}), \dots, \sigma_m(x_1^{(m)}), \dots, \sigma_m(x_{n_m}^{(m)}); k, \epsilon_0, P_1, \dots, P_{r_0}).$$

PROOF. We will prove the result by using the contradiction. Suppose, to the contrary, the result does not hold. Then there are some $r_0 \in \mathbb{N}$ and $\epsilon_0 > 0$ so that the following holds:

for any $r \in \mathbb{N}$, there are $k_r \in \mathbb{N}$ and

$$(A_1^{(i,r)}, \dots, A_{n_i}^{(i,r)}) \in \Gamma_R^{(top1/2)}(x_1^{(i)}, \dots, x_{n_i}^{(i)}; k_r, 1/r, P_1^{(i)}, \dots, P_r^{(i)}), \quad \text{for } 1 \leq i \leq m,$$

satisfying

$$(A_1^{(1,r)}, \dots, A_{n_1}^{(1,r)}, \dots, A_1^{(m,r)}, \dots, A_{n_m}^{(m,r)}) \notin \Gamma_R^{(top1/2)}(\sigma_1(x_1^{(1)}), \dots, \sigma_1(x_{n_1}^{(1)}), \dots, \sigma_m(x_1^{(m)}), \dots, \sigma_m(x_{n_m}^{(m)}); k_r, \epsilon_0, P_1, \dots, P_{r_0}). \quad (5.1)$$

Let γ be a free ultra-filter in $\beta(\mathbb{N}) \setminus \mathbb{N}$. Let $\prod_{r=1}^\gamma \mathcal{M}_{k_r}(\mathbb{C})$ be the C^* algebra ultra-product of matrices algebras $(\mathcal{M}_{k_r}(\mathbb{C}))_{r=1}^\infty$ along the ultra-filter γ , i.e. $\prod_{r=1}^\gamma \mathcal{M}_{k_r}(\mathbb{C})$ is the quotient algebra of the unital C^* -algebra $\prod_r^\infty \mathcal{M}_{k_r}(\mathbb{C})$ by \mathcal{I}_∞ , where $\mathcal{I}_\infty = \{(Y_r)_{r=1}^\infty \in \prod_r \mathcal{M}_{k_r}(\mathbb{C}) \mid \lim_{r \rightarrow \gamma} \|Y_r\| = 0\}$.

Let ϕ_i be the unital $*$ -homomorphism from the C^* -algebra \mathcal{A}_i into the C^* -algebra $\prod_{r=1}^\gamma \mathcal{M}_{k_r}(\mathbb{C})$, induced by the mapping

$$x_j^{(i)} \rightarrow [(A_j^{(i,r)})_r] \in \prod_{r=1}^\gamma \mathcal{M}_{k_r}(\mathbb{C}), \quad \forall 1 \leq j \leq n_i,$$

where $[(A_j^{(i,r)})_r]$ is the image of $(A_j^{(i,r)})_{r=1}^\infty$ in the quotient algebra $\prod_{r=1}^\gamma \mathcal{M}_{k_r}(\mathbb{C})$.

By the definition of full free product, we know that there is a unital $*$ -homomorphism ψ from \mathcal{D} into $\prod_{r=1}^\gamma \mathcal{M}_{k_r}(\mathbb{C})$ so that $\phi_i = \psi \circ \sigma_i$. Hence,

$$\begin{aligned} \lim_{r \rightarrow \gamma} \|P_t(A_1^{(1,r)}, \dots, A_{n_1}^{(1,r)}, \dots, A_1^{(m,r)}, \dots, A_{n_m}^{(m,r)})\| \\ \leq \|P_t(\sigma_1(x_1^{(1)}), \dots, \sigma_1(x_{n_1}^{(1)}), \dots, \sigma_m(x_1^{(m)}), \dots, \sigma_m(x_{n_m}^{(m)}))\|, \quad \forall 1 \leq t \leq r_0. \end{aligned}$$

This contradicts with the fact (5.1). This completes the proof of the lemma. \square

Recall the definition of a stable family of elements in a unital C^* -algebra in [19] as follows.

DEFINITION 5.2. *Suppose that \mathcal{A} is a unital C^* -algebra and x_1, \dots, x_n is a family of self-adjoint elements in \mathcal{A} . Suppose that $\{Q_r\}_{r=1}^\infty$ is a family of noncommutative polynomials in $\mathbb{C}\langle X_1, \dots, X_n \rangle$ with rational coefficients. The family of elements x_1, \dots, x_n is called stable if for any $\alpha < \delta_{top}(x_1, \dots, x_n)$ there are positive numbers $R, C > 0$ and $\omega_0 > 0$, $r_0 \geq 1$, $k_0 \geq 1$ so that*

$$\nu_\infty(\Gamma_R^{(top)}(x_1, \dots, x_n; q \cdot k_0, \frac{1}{r}, Q_1, \dots, Q_r), \omega) \geq C^{(q \cdot k_0)^2} \left(\frac{1}{\omega}\right)^{\alpha \cdot (q \cdot k_0)^2}, \quad \forall 0 < \omega < \omega_0, r > r_0, q \in \mathbb{N}.$$

EXAMPLE 5.1. *Any family of self-adjoint generators x_1, \dots, x_n of a finite dimensional C^* -algebra is stable. A self-adjoint element x in a unital C^* -algebra is stable. (see [19])*

5.2. Main result in this section. Now we are ready to show the additivity of topological free entropy dimension in the full free products of some unital C^* -algebras. In this subsection, we will use the following notation.

NOTATION 5.1. Suppose that $A \in \mathcal{M}_{k_1}(\mathbb{C})$ and $B \in \mathcal{M}_{k_2}(\mathbb{C})$. We denote the element

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \in \mathcal{M}_{k_1+k_2}(\mathbb{C})$$

by $A \oplus B$.

The concept of MF algebras was introduced by Blackadar and Kirchberg in [1]. It plays an important role in the classification of C^* -algebras and it is connected to the question whether the extension, in the sense of Brown, Douglas and Fillmore, of a unital C^* -algebra is a group (see the striking result of Haagerup and Thorbørnsen on $Ext(C_r^*(F_2))$).

The following result, which follows directly from Theorem 5.2 in the Appendix, shows that Voiculescu's topological free entropy dimension is well defined on a finite generated unital MF algebras.

LEMMA 5.3. Suppose that \mathcal{A} is a finitely generated unital Blackadar and Kirchberg's MF C^* -algebra. Then \mathcal{A} has the approximation property in the sense of Definition 4.2.

Now we are ready to prove our main result in this section.

THEOREM 5.1. Suppose that $\{\mathcal{A}_i\}_{i=1}^m$ ($m \geq 2$) is a family of unital C^* -algebras. Suppose that $\{x_j^{(i)}\}_{j=1}^{n_i}$ is a family of self-adjoint generators of \mathcal{A}_i for $i = 1, 2, \dots, m$. Suppose $\{x_j^{(i)}\}_{j=1}^{n_i}$ is a stable family in the sense of Definition 5.2 for $1 \leq i \leq m$. Let the unital C^* -algebra \mathcal{D} be the full free product of $\{\mathcal{A}_i\}_{i=1}^m$ equipped with the unital embedding $\{\sigma_i : \mathcal{A}_i \rightarrow \mathcal{D}\}_{i=1}^m$. Then

$$\delta_{top}(\sigma_1(x_1^{(1)}), \dots, \sigma_1(x_{n_1}^{(1)}), \dots, \sigma_m(x_1^{(m)}), \dots, \sigma_m(x_{n_m}^{(m)})) = \sum_{i=1}^m \delta_{top}(x_1^{(i)}, \dots, x_{n_i}^{(i)}).$$

If we identify each $x_j^{(i)}$ in \mathcal{A}_i with its image $\sigma_i(x_j^{(i)})$ in \mathcal{D} when no confusion arises, then

$$\delta_{top}(x_1^{(1)}, \dots, x_{n_1}^{(1)}, \dots, x_1^{(m)}, \dots, x_{n_m}^{(m)}) = \sum_{i=1}^m \delta_{top}(x_1^{(i)}, \dots, x_{n_i}^{(i)}).$$

PROOF. By Lemma 5.1, we need only to prove that

$$\delta_{top}(\sigma_1(x_1^{(1)}), \dots, \sigma_1(x_{n_1}^{(1)}), \dots, \sigma_m(x_1^{(m)}), \dots, \sigma_m(x_{n_m}^{(m)})) \geq \sum_{i=1}^m \delta_{top}(x_1^{(i)}, \dots, x_{n_i}^{(i)}).$$

Let $R > \max_{1 \leq i \leq m, 1 \leq j \leq n_i} \|x_j^{(i)}\|$ be a positive number. Since for all $1 \leq i \leq m$, $\{x_j^{(i)}\}_{j=1}^{n_i}$ is a stable family in the sense of Definition 5.2, each \mathcal{A}_i is a unital MF C^* algebras. By Theorem 5.4 in the appendix, the full free product \mathcal{D} is also an unital MF C^* -algebra. By Lemma 5.3, it follows that for any $r_0 \in \mathbb{N}$, there is some $k_0 \in \mathbb{N}$ so that

$$\Gamma_R^{(top)}(\sigma_1(x_1^{(1)}), \dots, \sigma_1(x_{n_1}^{(1)}), \dots, \sigma_m(x_1^{(m)}), \dots, \sigma_m(x_{n_m}^{(m)}); k_0, 1/r_0, P_1, \dots, P_{r_0}) \neq \emptyset.$$

Note that for all $1 \leq i \leq m$, $\{x_j^{(i)}\}_{j=1}^{n_i}$ is a stable family. By Definition 5.2, for all

$$\alpha_i < \delta_{top}(x_1^{(i)}, \dots, x_{n_i}^{(i)}), \quad i = 1, 2, \dots, m,$$

there are positive numbers $C > 0$ and $\omega_0 > 0$, $r_1 \geq r_0$, $k_1 \geq 1$ so that

$$\nu_\infty(\Gamma_R^{(top)}(x_1^{(i)}, \dots, x_{n_i}^{(i)}; q \cdot k_1, \frac{1}{r}, P_1^{(i)}, \dots, P_r^{(i)}), \omega) \geq C^{(q \cdot k_1)^2} \left(\frac{1}{\omega} \right)^{\alpha_i \cdot (q \cdot k_1)^2}, \quad (5.2)$$

for $0 < \omega < \omega_0$, $r > r_1$, $q \in \mathbb{N}$, and $1 \leq i \leq m$.

By Lemma 5.2, when r is large enough, for all $q \in \mathbb{N}$ we have that if

$$(A_1^{(1)}, \dots, A_{n_1}^{(1)}, \dots, A_1^{(m)}, \dots, A_{n_m}^{(m)}) \in \Gamma_R^{(top)}(\sigma_1(x_1^{(1)}), \dots, \sigma_1(x_{n_1}^{(1)}), \dots, \sigma_m(x_1^{(m)}), \dots, \sigma_m(x_{n_m}^{(m)}); k_0, 1/r_0, P_1, \dots, P_{r_0}) \quad (5.3)$$

and

$$(B_1^{(i)}, \dots, B_{n_i}^{(i)}) \in \Gamma_R^{(top1/2)}(x_1^{(i)}, \dots, x_{n_i}^{(i)}; qk_1, \epsilon, P_1^{(i)}, \dots, P_r^{(i)}), \quad \text{for } 1 \leq i \leq m, \quad (5.4)$$

then

$$(A_1^{(1)} \oplus B_1^{(1)}, \dots, A_{n_1}^{(1)} \oplus B_{n_1}^{(1)}, \dots, A_1^{(m)} \oplus B_1^{(m)}, \dots, A_{n_m}^{(m)} \oplus B_{n_m}^{(m)}) \in \Gamma_R^{(top)}(\sigma_1(x_1^{(1)}), \dots, \sigma_1(x_{n_1}^{(1)}), \dots, \sigma_m(x_1^{(m)}), \dots, \sigma_m(x_{n_m}^{(m)}); k_0 + qk_1, 1/r_0, P_1, \dots, P_{r_0}). \quad (5.5)$$

On the other hand, by (5.2), for $0 < \omega < \omega_0/2$, r large enough,

$$\begin{aligned} \nu_\infty(\Gamma_R^{(top1/2)}(x_1^{(i)}, \dots, x_{n_i}^{(i)}; qk_1, 1/r, P_1^{(i)}, \dots, P_r^{(i)}), 2\omega) \\ \geq \nu_\infty(\Gamma_R^{(top)}(x_1^{(i)}, \dots, x_{n_i}^{(i)}; qk_1, \frac{1}{r}, P_1^{(i)}, \dots, P_r^{(i)}), 2\omega) \\ \geq C^{(q \cdot k_1)^2} \left(\frac{1}{2\omega} \right)^{\alpha_i \cdot (q \cdot k_1)^2}, \quad \forall q \in \mathbb{N}. \end{aligned} \quad (5.6)$$

Hence by the relationship between packing number and covering number, we know there is a family of elements

$$\{(B_1^{(\lambda_i)}, \dots, B_{n_i}^{(\lambda_i)})\}_{\lambda_i \in \Lambda_i} \subset \Gamma_R^{(top1/2)}(x_1^{(i)}, \dots, x_{n_i}^{(i)}; qk_1, \epsilon, P_1^{(i)}, \dots, P_r^{(i)}), \quad \text{for } 1 \leq i \leq m, \quad (5.7)$$

where Λ_i is an index set satisfying

$$|\Lambda_i| \geq \nu_\infty(\Gamma_R^{(top1/2)}(x_1^{(i)}, \dots, x_{n_i}^{(i)}; qk_1, 1/r, P_1^{(i)}, \dots, P_r^{(i)}), 2\omega) \geq C^{(q \cdot k_1)^2} \left(\frac{1}{2\omega} \right)^{\alpha_i \cdot (q \cdot k_1)^2}; \quad (5.8)$$

and

$$\|(B_1^{(\lambda_i)}, \dots, B_{n_i}^{(\lambda_i)}) - (B_1^{(\tilde{\lambda}_i)}, \dots, B_{n_i}^{(\tilde{\lambda}_i)})\| \geq \omega, \quad \forall \lambda_i \neq \tilde{\lambda}_i \in \Lambda_i. \quad (5.9)$$

Thus, combining (5.3), (5.4), (5.5), (5.7), (5.8) and (5.9) we have

$$\begin{aligned} \nu_\infty(\Gamma_R^{(top)}(\sigma_1(x_1^{(1)}), \dots, \sigma_1(x_{n_1}^{(1)}), \dots, \sigma_m(x_1^{(m)}), \dots, \sigma_m(x_{n_m}^{(m)}); k_0 + qk_1, 1/r_0, P_1, \dots, P_{r_0}), \omega) \\ \geq \prod_{i=1}^m |\Lambda_i| \geq \prod_{i=1}^m \left(C^{(q \cdot k_1)^2} \left(\frac{1}{2\omega} \right)^{\alpha_i \cdot (q \cdot k_1)^2} \right), \quad \forall q \in \mathbb{N}. \end{aligned}$$

This induces that

$$\delta_{top}(\sigma_1(x_1^{(1)}), \dots, \sigma_1(x_{n_1}^{(1)}), \dots, \sigma_m(x_1^{(m)}), \dots, \sigma_m(x_{n_m}^{(m)})) \geq \sum_{i=1}^m \alpha_i,$$

whence

$$\delta_{top}(\sigma_1(x_1^{(1)}), \dots, \sigma_1(x_{n_1}^{(1)}), \dots, \sigma_m(x_1^{(m)}), \dots, \sigma_m(x_{n_m}^{(m)})) \geq \sum_{i=1}^m \delta_{top}(x_1^{(i)}, \dots, x_{n_i}^{(i)}).$$

By Lemma 5.1, we get

$$\delta_{top}(\sigma_1(x_1^{(1)}), \dots, \sigma_1(x_{n_1}^{(1)}), \dots, \sigma_m(x_1^{(m)}), \dots, \sigma_m(x_{n_m}^{(m)})) = \sum_{i=1}^m \delta_{top}(x_1^{(i)}, \dots, x_{n_i}^{(i)}).$$

Or

$$\delta_{top}(x_1^{(1)}, \dots, x_{n_1}^{(1)}, \dots, x_1^{(m)}, \dots, x_{n_m}^{(m)}) = \sum_{i=1}^m \delta_{top}(x_1^{(i)}, \dots, x_{n_i}^{(i)}),$$

when no confusion arises. \square

As a corollary, we have the following result.

COROLLARY 5.1. *Suppose that \mathcal{A}_i ($i = 1, 2, \dots, m$) is a unital C^* algebra generated by a self-adjoint element x_i in \mathcal{A}_i . Let \mathcal{D} be the full free product of $\mathcal{A}_1, \dots, \mathcal{A}_n$ equipped with unital embedding from each \mathcal{A}_i into \mathcal{D} . Identify the element x_i in \mathcal{A}_i with its image in \mathcal{D} . Then*

$$\delta_{top}(x_1, \dots, x_n) = \sum_{i=1}^n \delta_{top}(x_i) = n - \sum_{i=1}^n \frac{1}{n_i},$$

where n_i is the number of elements in the spectrum of x_i in \mathcal{A}_i . (We use the notation $1/\infty = 0$)

PROOF. It follows from Example 5.1, Theorem 5.1 and the results in [18]. \square

COROLLARY 5.2. *Suppose that \mathcal{A}_i is a finite dimensional C^* -algebra generated by a family of self-adjoint element $\{x_j^{(i)}\}_{1 \leq j \leq n_i}$ for $1 \leq i \leq n$. Let \mathcal{D} be the full free product of $\mathcal{A}_1, \dots, \mathcal{A}_n$ equipped with unital embedding from each \mathcal{A}_i into \mathcal{D} . Identify the element $\{x_j^{(i)}\}$ in \mathcal{A}_i with its image in \mathcal{D} . Then*

$$\delta_{top}(\{x_j^{(i)}\}_{1 \leq j \leq n_i, 1 \leq i \leq n}) = \sum_{i=1}^n \delta_{top}(\{x_j^{(i)}\}_{1 \leq j \leq n_i}) = n - \sum_{i=1}^n \frac{1}{\dim_{\mathbb{C}} \mathcal{A}_i},$$

where $\dim_{\mathbb{C}} \mathcal{A}_i$ is the complex dimension of \mathcal{A}_i .

PROOF. It follows from Example 5.1, Theorem 5.1 and the results in [19]. \square

Appendix: Full Free Product of Unital MF C*-algebras is MF Algebra

The concept of MF algebras was introduced by Blackadar and Kirchberg in [1]. This class of C*-algebras is of interest for many reasons. For example, it plays an important role in the classification of C*-algebras and it is connected to the question whether the extension semigroup, in the sense of Brown, Douglas and Fillmore, of a unital C*-algebra is a group (see the striking result of Haagerup and Thorbørnsen on $Ext(C_r^*(F_2))$). Thanks to Voiculescu's result in [32], we know that every quasidigonal C*-algebra is an MF algebra. Many properties of MF algebras have been discussed in [1]. For example, it was shown there that the inductive limit of MF algebras is an MF algebra and every subalgebra of an MF algebra is an MF algebra. In this appendix, we will prove that *unital full free product of two unital separable MF algebras is, again, an MF algebra*.

Let us fix notation first. We always assume that \mathcal{H} is a separable complex Hilbert space and $B(\mathcal{H})$ is the set of all bounded operators on \mathcal{H} . Suppose $\{x, x_k\}_{k=1}^\infty$ is a family of elements in $B(\mathcal{H})$. We say $x_k \rightarrow x$ in *-SOT (*-strong operator topology) if and only if $x_k \rightarrow x$ in SOT and $x_k^* \rightarrow x^*$ in SOT. Suppose $\{x_1, \dots, x_n\}$ and $\{x_1^{(k)}, \dots, x_n^{(k)}\}_{k=1}^\infty$ are families of elements in $B(\mathcal{H})$. We say

$$\langle x_1^{(k)}, \dots, x_n^{(k)} \rangle \rightarrow \langle x_1, \dots, x_n \rangle, \text{ in } *-SOT, \text{ as } k \rightarrow \infty$$

if and only if

$$x_i^{(k)} \rightarrow x_i \text{ in } *-SOT, \text{ as } k \rightarrow \infty, \quad \forall 1 \leq i \leq n.$$

Suppose $\{\mathcal{A}_k\}_{k=1}^\infty$ is a family of unital C*-algebras. Let $\prod \mathcal{A}_k$ be C*-direct product of the \mathcal{A}_k , i.e. the set of bounded sequences $(x_k)_{k=1}^\infty$, with $x_k \in \mathcal{A}_k$, with pointwise operations and sup norm; and let $\sum \mathcal{A}_k$ be the C*-direct sum, the set of sequences converging to zero in norm. Then $\prod \mathcal{A}_k$ is a C*-algebra and $\sum \mathcal{A}_k$ is a closed two-sided ideal; let π be the quotient map from $\prod \mathcal{A}_k$ to $\prod \mathcal{A}_k / \sum \mathcal{A}_k$. Then $\prod \mathcal{A}_k / \sum \mathcal{A}_k$ is a unital C*-algebra. If we denote $\pi((x_k)_{k=1}^\infty)$ by $[(x_k)_k]$ for any $(x_k)_{k=1}^\infty$ in $\prod \mathcal{A}_k$, then

$$\|[(x_k)_k]\| = \limsup_{k \rightarrow \infty} \|x_k\|.$$

Suppose \mathcal{A} is a separable unital C*-algebra on a Hilbert space \mathcal{H} . Let $\mathcal{H}^\infty = \bigoplus_{\mathbb{N}} \mathcal{H}$, and for any $x \in \mathcal{A}$, let x^∞ be the element $\bigoplus_{\mathbb{N}} x = (x, x, x, \dots)$ in $\prod \mathcal{A}^{(k)} \subset B(\mathcal{H}^\infty)$, where $\mathcal{A}^{(k)}$ is the k -th copy of \mathcal{A} .

Suppose \mathcal{A} is a separable unital C*-algebra and $\pi_i : \mathcal{A} \rightarrow B(\mathcal{H}_i)$ are unital *-representations for $i = 1, 2$. If there is a unitary $U : \mathcal{H}_1 \rightarrow \mathcal{H}_2$, $n = 1, 2, \dots$, satisfying

$$U^* \pi_2(x) U = \pi_1(x), \quad \forall x \in \mathcal{A},$$

then we say that π_1 and π_2 are unitarily equivalent, which is denoted by $\pi_1 \approx_u \pi_2$. Furthermore, if \mathcal{A} is generated by a family of self-adjoint elements $\{x_1, \dots, x_n\}$, then $\pi_1 \approx_u \pi_2$ will also be

denoted by

$$\langle \pi_1(x_1), \dots, \pi_1(x_n) \rangle \approx_u \langle \pi_2(x_1), \dots, \pi_2(x_n) \rangle.$$

Suppose \mathcal{A} is a separable unital C^* -algebra and $\pi_i : \mathcal{A} \rightarrow B(\mathcal{H}_i)$ are unital $*$ -representations for $i = 1, 2$. If there is a sequence of unitaries $U_n : \mathcal{H}_1 \rightarrow \mathcal{H}_2$, $n = 1, 2, \dots$, satisfying

$$\|U_n^* \pi_2(x) U_n - \pi_1(x)\| \rightarrow 0, \quad \text{as } n \rightarrow \infty, \quad \forall x \in \mathcal{A},$$

then we say that π_1 and π_2 are approximately unitarily equivalent, which is denoted by $\pi_1 \sim_a \pi_2$. Recall the following important result by Voiculescu:

LEMMA 5.4. *Let \mathcal{A} be a separable unital C^* -algebra and $\pi_i : \mathcal{A} \rightarrow B(\mathcal{H}_i)$ be unital faithful $*$ -representations for $i = 1, 2$ satisfying $\pi_i(\mathcal{A}) \cap \mathcal{K}(\mathcal{H}_i) = 0$ for $i = 1, 2$, where $\mathcal{K}(\mathcal{H}_i)$ is the set of compact operators on \mathcal{H}_i . Then $\pi_1 \sim_a \pi_2$.*

The following lemma will be needed in the proof of Theorem 5.2.

LEMMA 5.5. *Suppose that \mathcal{H}_1 and \mathcal{H}_2 are two separable infinite dimensional complex Hilbert spaces. Let $\{e_n\}_{n=1}^\infty$ be an orthonormal basis of \mathcal{H}_1 and $\{f_n\}_{n=1}^\infty$ be an orthonormal basis of \mathcal{H}_2 . (Thus $\{e_n \oplus 0, 0 \oplus f_m\}_{m,n=1}^\infty$ is an orthonormal basis of $\mathcal{H}_1 \oplus \mathcal{H}_2$). Let*

$$T = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in B(\mathcal{H}_1 \oplus \mathcal{H}_2),$$

where $A \in B(\mathcal{H}_1)$, $B \in B(\mathcal{H}_2, \mathcal{H}_1)$, $C \in B(\mathcal{H}_1, \mathcal{H}_2)$ and $D \in B(\mathcal{H}_2)$. Assume $\{U_n\}_{n=1}^\infty$ is a family of unitary operators from \mathcal{H}_2 to $\mathcal{H}_1 \oplus \mathcal{H}_2$ so that

$$U_n(f_i) = 0 \oplus f_i, \quad \forall 1 \leq i \leq n.$$

Then the following statements are true:

- (a) $U_n^* T U_n \rightarrow D$ in the weak operator topology.
- (b) If $B = 0$ and $C = 0$, then $U_n^* T U_n \rightarrow D$ in the $*$ -strong operator topology.

PROOF. It is an easy exercise. □

The following results gives some equivalent definitions of an MF algebra.

THEOREM 5.2. *Suppose that \mathcal{A} is a unital C^* -algebra generated by a family of self-adjoint elements x_1, \dots, x_n in \mathcal{A} . Then the following are equivalent:*

- (1) \mathcal{A} is an MF algebra, i.e. there is a unital embedding from \mathcal{A} into $\prod_k \mathcal{M}_{n_k}(\mathbb{C}) / \sum \mathcal{M}_{n_k}(\mathbb{C})$ for a sequence of positive integers $\{n_k\}_{k=1}^\infty$.
- (2) \mathcal{A} has approximation property in the sense of Definition 4.2;
- (3) There are a sequence of positive integers $\{m_k\}_{k=1}^\infty$ and self-adjoint matrices $A_1^{(k)}, \dots, A_n^{(k)}$ in $\mathcal{M}_{m_k}^{s.a.}(\mathbb{C})$ for $k = 1, 2, \dots$, such that

$$\lim_{k \rightarrow \infty} \|P(A_1^{(k)}, \dots, A_n^{(k)})\| = \|P(x_1, \dots, x_n)\|, \quad \forall P \in \mathbb{C}\langle X_1, \dots, X_n \rangle,$$

where $\mathbb{C}\langle X_1, \dots, X_n \rangle$ is the set of all noncommutative polynomials in the indeterminates X_1, \dots, X_n .

- (4) Suppose $\pi : \mathcal{A} \rightarrow B(\mathcal{H})$ is a faithful $*$ -representation of \mathcal{A} on an infinite dimensional separable complex Hilbert space \mathcal{H} . Then there are a sequence of positive integers $\{m_k\}_{k=1}^\infty$, families of self-adjoint matrices $\{A_1^{(k)}, \dots, A_n^{(k)}\}$ in $\mathcal{M}_{m_k}^{s.a.}(\mathbb{C})$ for $k = 1, 2, \dots$, and unitary operators $U_k : \mathcal{H} \rightarrow (\mathbb{C}^{m_k})^\infty$ for $k = 1, 2, \dots$, such that
- (a)

$$\lim_{k \rightarrow \infty} \|P(A_1^{(k)}, \dots, A_n^{(k)})\| = \|P(x_1, \dots, x_n)\|, \quad \forall P \in \mathbb{C}\langle X_1, \dots, X_n \rangle;$$

(b)

$$U_k^* (A_i^{(k)})^\infty U_k \rightarrow \pi(x_i) \text{ in } *-SOT \text{ as } k \rightarrow \infty, \quad \text{for } 1 \leq i \leq n,$$

$$\text{where } (A_i^{(k)})^\infty = A_i^{(k)} \oplus A_i^{(k)} \oplus A_i^{(k)} \dots \in B((\mathbb{C}^{m_k})^\infty).$$

PROOF. (1) \Leftrightarrow (2) \Leftrightarrow (3) are directly from Theorem 3.2.2 in [1], Definition 5.3 and Lemma 5.6 in [18]. (4) \Rightarrow (3) is trivial. We need only to prove (3) \Rightarrow (4).

Let $\{P_r\}_{r=1}^\infty$ be the collection of all noncommutative polynomials in $\mathbb{C}\langle X_1, \dots, X_n \rangle$ with rational complex coefficients. Let $\{\xi_r\}_{r=1}^\infty$ be a dense subset of the unit ball in \mathcal{H} . To show that (3) \Rightarrow (4), we need only to prove the following: Let $\pi : \mathcal{A} \rightarrow B(\mathcal{H})$ be a faithful $*$ -representation of \mathcal{A} on an infinite dimensional separable complex Hilbert space \mathcal{H} . For any positive integer k , there are a positive integer m_k , a family of self-adjoint matrices $\{A_1^{(k)}, \dots, A_n^{(k)}\}$ in $\mathcal{M}_{m_k}^{s.a.}(\mathbb{C})$, and a unitary operator $U_k : \mathcal{H} \rightarrow (\mathbb{C}^{m_k})^\infty$, such that

(a)

$$\| \|P_r(A_1^{(k)}, \dots, A_n^{(k)})\| - \|P_r(x_1, \dots, x_n)\| \| < \frac{1}{k}, \quad \forall 1 \leq r \leq k;$$

(b)

$$\|U_k^* (A_i^{(k)})^\infty U_k \cdot \xi_r - \pi(x_i) \cdot \xi_r\| < \frac{1}{k}, \quad \text{for } 1 \leq i \leq n, \quad 1 \leq r \leq k.$$

Assume that (3) is true. Thus there are a sequence of positive integers $\{t_s\}_{s=1}^\infty$ and families of self-adjoint matrices $\{B_1^{(s)}, \dots, B_n^{(s)}\}$ in $\mathcal{M}_{t_s}^{s.a.}(\mathbb{C})$ for $s = 1, 2, \dots$, such that

$$\lim_{s \rightarrow \infty} \|P(B_1^{(s)}, \dots, B_n^{(s)})\| = \|P(x_1, \dots, x_n)\|, \quad \forall P \in \mathbb{C}\langle X_1, \dots, X_n \rangle.$$

For any positive integers $N_1 < N_2$ and $1 \leq i \leq n$, we define

$$\begin{aligned} D(N_1, N_2; i) &= B_i^{(N_1)} \oplus B_i^{(N_1+1)} \oplus \dots \oplus B_i^{(N_2)} \\ D(N_1; i) &= B_i^{(N_1)} \oplus B_i^{(N_1+1)} \oplus B_i^{(N_1+2)} \oplus \dots \\ D(N_1, N_2; i)^\infty &= D(N_1, N_2; i) \oplus D(N_1, N_2; i) \oplus D(N_1, N_2; i) \oplus \dots \\ D(N_1; i)^\infty &= D(N_1; i) \oplus D(N_1; i) \oplus D(N_1; i) \oplus \dots \end{aligned}$$

It is not hard to see that, for any $P \in \mathbb{C}\langle X_1, \dots, X_n \rangle$,

$$\|P(D(N_1, N_2; 1)^\infty, \dots, D(N_1, N_2; n)^\infty)\| = \|P(D(N_1, N_2; 1), \dots, D(N_1, N_2; n))\| \quad (5.10)$$

$$\|P(D(N_1; 1)^\infty, \dots, D(N_1; n)^\infty)\| = \|P(D(N_1; 1), \dots, D(N_1; n))\| \quad (5.11)$$

$$\lim_{N_2 \rightarrow \infty} \|P(D(N_1, N_2; 1), \dots, D(N_1, N_2; n))\| = \|P(D(N_1; 1), \dots, D(N_1; n))\| \quad (5.12)$$

$$\begin{aligned} \|P(D(N_1; 1)^\infty, \dots, D(N_1; n)^\infty)\| &= \sup_{N_2 \geq N_1} \|P(B_1^{(N_2)}, \dots, B_n^{(N_2)})\| \\ &\geq \|P(x_1, \dots, x_n)\| \end{aligned} \quad (5.13)$$

$$\lim_{N_1 \rightarrow \infty} \|P(D(N_1; 1), \dots, D(N_1; n))\| = \|P(x_1, \dots, x_n)\|. \quad (5.14)$$

Furthermore, we let

$$\begin{aligned} E(N_1, N_2; i) &= (B_i^{(N_1)})^\infty \oplus (B_i^{(N_1+1)})^\infty \oplus (B_i^{(N_1+2)})^\infty \oplus \dots \oplus (B_i^{(N_2)})^\infty \\ E(N_1; i) &= (B_i^{(N_1)})^\infty \oplus (B_i^{(N_1+1)})^\infty \oplus (B_i^{(N_1+2)})^\infty \oplus \dots \end{aligned}$$

It is easy to see that

$$\langle E(N_1, N_2; 1), \dots, E(N_1, N_2; n) \rangle \approx_u \langle D(N_1, N_2; 1)^\infty, \dots, D(N_1, N_2; n)^\infty \rangle \quad (5.15)$$

$$\langle E(N_1; 1), \dots, E(N_1; n) \rangle \approx_u \langle D(N_1; 1)^\infty, \dots, D(N_1; n)^\infty \rangle. \quad (5.16)$$

Let $\mathcal{D}(N_1) = C^*(D(N_1; 1)^\infty, \dots, D(N_1; n)^\infty)$ be the unital C^* -algebra generated by

$$D(N_1; 1)^\infty, \dots, D(N_1; n)^\infty$$

on a separable Hilbert space \mathcal{H}_1 . Let $id_{\mathcal{D}(N_1)}$ be the identity representation of $\mathcal{D}(N_1)$ on \mathcal{H}_1 , i.e.

$$id_{\mathcal{D}(N_1)} : \mathcal{D}(N_1) \rightarrow \mathcal{D}(N_1)$$

is defined by $id_{\mathcal{D}(N_1)}(x) = x$ for any x in $\mathcal{D}(N_1)$. By the inequality (5.13), we know there is a unital $*$ -homomorphism

$$\rho_{N_1} : \mathcal{D}(N_1) \rightarrow \pi(\mathcal{A})$$

such that

$$\rho_{N_1}(D(N_1; i)^\infty) = \pi(x_i), \quad \forall 1 \leq i \leq n,$$

because π is a faithful $*$ -representation of \mathcal{A} on \mathcal{H} . Denote

$$\begin{aligned} \vec{D}(N_1) &= \langle D(N_1; 1)^\infty, \dots, D(N_1; n)^\infty \rangle \\ \vec{x} &= \langle x_1, \dots, x_n \rangle. \end{aligned}$$

By Lemma 5.4, we know that

$$id_{\mathcal{D}(N_1)} \sim_a id_{\mathcal{D}(N_1)} \oplus \rho_{N_1}.$$

It follows that,

$$\begin{aligned}\vec{D}(N_1) \oplus \pi(\vec{x}) &= \langle D(N_1; 1)^\infty \oplus \pi(x_1), \dots, D(N_1; n)^\infty \oplus \pi(x_n) \rangle \\ &= (id_{\mathcal{D}(N_1)} \oplus \rho_{N_1})(\vec{D}(N_1)) \\ &\in \overline{\left\{ W^* \vec{D}(N_1) W \mid W : \mathcal{H}_1 \oplus \mathcal{H} \rightarrow \mathcal{H}_1 \text{ is a unitary operator} \right\}}^{Norm}.\end{aligned}$$

By Lemma 5.5, we know that

$$\pi(\vec{x}) \in \overline{\left\{ V^* \vec{D}(N_1) \oplus \pi(\vec{x}) V \mid V : \mathcal{H} \rightarrow \mathcal{H}_1 \oplus \mathcal{H} \text{ is a unitary operator} \right\}}^{*-SOT}.$$

It induces that

$$\pi(\vec{x}) \in \overline{\left\{ U_1^* \vec{D}(N_1) U_1 \mid U_1 : \mathcal{H} \rightarrow \mathcal{H}_1 \text{ is a unitary operator} \right\}}^{*-SOT}. \quad (5.17)$$

Take a positive integer k . For such k and the family of polynomials $\{P_1, \dots, P_k\}$, by equation (5.11), equation (5.14), we know that there is a positive integer N_1 such that

(i)

$$|||P_r(D(N_1; 1), \dots, D(N_1; n))|| - ||P_r(x_1, \dots, x_n)||| < \frac{1}{4k}, \quad \forall 1 \leq r \leq k.$$

For the family of vectors $\{\xi_1, \dots, \xi_k\}$, by the fact (5.17) we know there is a unitary $U_1 : \mathcal{H} \rightarrow \mathcal{H}_1$, such that

(ii)

$$||U_1^* D(N_1; i)^\infty U_1 \cdot \xi_r - \pi(x_i) \cdot \xi_r|| < \frac{1}{4k}, \quad \text{for } 1 \leq i \leq n, \quad 1 \leq r \leq k.$$

In view of the fact (5.16), i.e.

$$\langle D(N_1; 1)^\infty, \dots, D(N_1; n)^\infty \rangle \approx_u \langle E(N_1; 1), \dots, E(N_1; n) \rangle$$

and the fact (5.11), together with the definitions of $E(N_1, N_2; i)$ and $E(N_1; i)$, we know there are a positive integer $N_2 > N_1$ and a unitary $U_2 : \mathcal{H} \rightarrow \mathcal{H}_3$ such that

(iii)

$$|||P_r(E(N_1, N_2; 1), \dots, E(N_1, N_2; n))|| - ||P_r(x_1, \dots, x_n)||| < \frac{1}{2k}, \quad \forall 1 \leq r \leq k;$$

(iv)

$$||U_2^* E(N_1, N_2; i) U_2 \cdot \xi_r - \pi(x_i) \cdot \xi_r|| < \frac{1}{2k}, \quad \text{for } 1 \leq i \leq n, \quad 1 \leq r \leq k,$$

where we assume that $E(N_1, N_2; 1), \dots, E(N_1, N_2; n)$ act on a separable Hilbert space \mathcal{H}_3 . In view of the fact (5.15), i.e.

$$\langle E(N_1, N_2; 1), \dots, E(N_1, N_2; n) \rangle \approx_u \langle D(N_1, N_2; 1)^\infty, \dots, D(N_1, N_2; n)^\infty \rangle,$$

if we let

$$A_i^{(k)} = D(N_1, N_2; i) \in \mathcal{M}_{m_k}^{s.a.}(\mathbb{C}), \quad 1 \leq i \leq n$$

for some $m_k \in \mathbb{N}$, then there is some $U_k : \mathcal{H} \rightarrow (\mathbb{C}^{m_k})^\infty$ such that

(a)

$$|||P_r(A_1^{(k)}, \dots, A_n^{(k)})|| - ||P_r(x_1, \dots, x_n)||| < \frac{1}{k}, \quad \forall 1 \leq r \leq k;$$

(b)

$$||U_k^* (A_i^{(k)})^\infty U_k \cdot \xi_r - \pi(x_i) \cdot \xi_r|| < \frac{1}{k}, \quad \text{for } 1 \leq i \leq n, 1 \leq r \leq k.$$

This completes the proof of the theorem. \square

Recall that $\mathcal{A} \subset B(\mathcal{H})$ is a separable quasidiagonal C^* -algebra if there is an increasing sequence of finite-rank projections $\{E_i\}_{i=1}^\infty$ on H tending strongly to the identity such that $\|xE_i - E_ix\| \rightarrow 0$ as $i \rightarrow \infty$ for any $x \in \mathcal{A}$. The examples of quasidiagonal C^* -algebras include all abelian C^* -algebra and all finite dimensional C^* -algebras. An abstract separable C^* -algebra \mathcal{A} is quasidiagonal if there is a faithful $*$ -representation $\pi : \mathcal{A} \rightarrow B(\mathcal{H})$ such that $\pi(\mathcal{A}) \subset B(\mathcal{H})$ is quasidiagonal.

Following lemma will be needed in the proof of Theorem 5.3.

LEMMA 5.6. *Suppose that $\mathcal{A} \subset B(\mathcal{H})$ is a separable unital quasidiagonal C^* -algebra, and x_1, \dots, x_n are self-adjoint elements in \mathcal{A} . For any $\epsilon > 0$, any finite subset $\{P_1, \dots, P_r\}$ of $\mathbb{C}\langle X_1, \dots, X_n \rangle$ and any finite subset $\{\xi_1, \dots, \xi_r\}$ of \mathcal{H} , there is a finite rank projection p in $B(\mathcal{H})$ such that:*

- (i) $\|\xi_k - p \xi_k\| < \epsilon$, $\|p x_i p \xi_k - x_i \xi_k\| < \epsilon$, for all $1 \leq i \leq n$ and $1 \leq k \leq r$;
- (ii) $|||P_j(p x_1 p, \dots, p x_n p)||_{B(p\mathcal{H})} - ||P_j(x_1, \dots, x_n)||| < \epsilon$, for all $1 \leq j \leq r$.

PROOF. It follows directly from the definition of quasidiagonality of a separable C^* -algebra. \square

THEOREM 5.3. *Suppose that \mathcal{H} is a separable complex Hilbert space, and $r \in \mathbb{N}$. Suppose that*

$$\{P_1, \dots, P_r\} \subset \mathbb{C}\langle X_1, \dots, X_n \rangle; \quad \{Q_1, \dots, Q_r\} \subset \mathbb{C}\langle Y_1, \dots, Y_m \rangle$$

are families of noncommutative polynomials. Suppose that

$$\{\xi_1, \dots, \xi_r\} \subset \mathcal{H}$$

is a family of unit vectors in \mathcal{H} . Suppose that $\mathcal{A} \subset B(\mathcal{H})$ and $\mathcal{B} \subset B(\mathcal{H})$ are two separable unital quasidiagonal C^ -algebras on \mathcal{H} . Assume that \mathcal{A} , or \mathcal{B} , is generated by a family of self-adjoint elements x_1, \dots, x_n in \mathcal{A} , or by a family of self-adjoint elements y_1, \dots, y_m in \mathcal{B} respectively. Then there are a positive integer t , a family of self-adjoint matrices*

$$\{A_1, \dots, A_n, B_1, \dots, B_m\} \subset \mathcal{M}_t^{s.a.}(\mathbb{C}),$$

and unitary $U : \mathcal{H} \rightarrow (\mathbb{C}^t)^\infty$ such hat

(1)

$$|||P_i(A_1, \dots, A_n)|| - ||P_i(x_1, \dots, x_n)||| < \frac{1}{r}; \quad \forall 1 \leq i \leq r$$

and

$$(2) \quad |||Q_i(B_1, \dots, B_m)|| - ||Q_i(y_1, \dots, y_m)||| < \frac{1}{r}; \quad \forall 1 \leq i \leq r$$

$$\|U^* A_i^\infty U \xi_k - x_i \xi_k\| < \frac{1}{r}, \quad \forall 1 \leq i \leq n, 1 \leq k \leq r;$$

and

$$\|U^* B_j^\infty U \xi_k - y_j \xi_k\| < \frac{1}{r}, \quad \forall 1 \leq j \leq m, 1 \leq k \leq r,$$

where we follow the previous notation by assuming

$$A_i^\infty = A_i \oplus A_i \oplus \dots; \quad B_j^\infty = B_j \oplus B_j \oplus \dots$$

are operators acting on the Hilbert space $(\mathbb{C}^t)^\infty = \mathbb{C}^t \oplus \mathbb{C}^t \oplus \mathbb{C}^t \oplus \dots$.

PROOF. Without the loss of generality, we will assume that $\|x_i\| \leq 1$ and $\|y_j\| \leq 1$ for all $1 \leq i \leq n, 1 \leq j \leq m$. Note that \mathcal{A} and \mathcal{B} are two separable unital quasidiagonal C^* -algebras on \mathcal{H} . By Lemma 5.6, there are finite rank projections p and q in $B(\mathcal{H})$ such that:

(i) for all $1 \leq i \leq n, 1 \leq j \leq m$ and $1 \leq k \leq r$, we have

$$\|p \xi_k - \xi_k\| < \frac{1}{2r},$$

and

$$\|q \xi_k - \xi_k\| < \frac{1}{2r};$$

(ii) for all $1 \leq i \leq r$, we have

$$|||P_i(p x_1 p, \dots, p x_n p)|| - ||P_i(x_1, \dots, x_n)||| < \frac{1}{r},$$

and

$$|||Q_i(q y_1 q, \dots, q y_m q)|| - ||Q_i(y_1, \dots, y_m)||| < \frac{1}{r}.$$

Let \tilde{p} be a finite rank projection in $B(\mathcal{H})$ such that $p \leq \tilde{p}$, $q \leq \tilde{p}$ and $\text{rank}(\tilde{p})$ is a common multiple of $\text{rank}(p)$ and $\text{rank}(q)$. Therefore there are equivalent mutually orthogonal projections $e_1 = p, e_2, \dots, e_{d_1}$ and equivalent mutually orthogonal projections $f_1 = q, f_2, \dots, f_{d_2}$ such that $\sum_{i=1}^{d_1} e_i = \sum_{j=1}^{d_2} f_j = \tilde{p}$. Let v_1, \dots, v_{d_1} and w_1, \dots, w_{d_2} be partial isometries in $B(\mathcal{H})$ such that $v_i^* v_i = e_i = p, v_i v_i^* = e_i, w_j^* w_j = f_j = q$, and $w_j w_j^* = f_j$. Let $\mathcal{K} = \tilde{p} \mathcal{H}$ and

$$\begin{aligned} \tilde{x}_i &= \sum_{i_1=1}^{d_1} v_{i_1} x_i v_{i_1}^*, \quad 1 \leq i \leq n \\ \tilde{y}_j &= \sum_{j_1=1}^{d_2} w_{j_1} y_j w_{j_1}^*, \quad 1 \leq j \leq m. \end{aligned}$$

It is easy to check that

(iii) for all $1 \leq i \leq n$, $1 \leq j \leq m$ and $1 \leq k \leq r$, we have

$$\|\tilde{x}_i p \xi_k - x_i \xi_k\| \leq \|\tilde{x}_i p \xi_k - x_i p \xi_k\| + \|x_i p \xi_k - x_i \xi_k\| < \frac{1}{r},$$

and

$$\|\tilde{y}_j q \xi_k - y_j \xi_k\| \leq \|\tilde{y}_j q \xi_k - y_j q \xi_k\| + \|y_j q \xi_k - y_j \xi_k\| < \frac{1}{r};$$

(iv) for all $1 \leq i \leq r$, we have

$$|||P_i(\tilde{x}_1, \dots, \tilde{x}_n)|| - |||P_i(x_1, \dots, x_n)||| = |||P_i(p x_1 p, \dots, p x_n p)|| - |||P_i(x_1, \dots, x_n)||| < \frac{1}{r},$$

and

$$|||Q_i(\tilde{y}_1, \dots, \tilde{y}_m)|| - |||Q_i(y_1, \dots, y_m)||| = |||Q_i(q y_1 q, \dots, q y_m q)|| - |||Q_i(y_1, \dots, y_m)||| < \frac{1}{r}.$$

Let $t = \dim_{\mathbb{C}} \mathcal{K}$ and $\rho : B(\mathcal{K}) \rightarrow \mathcal{M}_t(\mathbb{C})$ be a $*$ -isomorphism from $B(\mathcal{K})$ onto $\mathcal{M}_t(\mathbb{C})$. Let

$$A_i = \rho(\tilde{x}_i), \quad B_j = \rho(\tilde{y}_j) \quad \text{for } 1 \leq i \leq n, \quad 1 \leq j \leq m;$$

and

$$A_i^\infty = A_i \oplus A_i \oplus \dots; \quad B_j^\infty = B_j \oplus B_j \oplus \dots$$

be operators acting on the Hilbert space $(\mathbb{C}^t)^\infty = \mathbb{C}^t \oplus \mathbb{C}^t \oplus \mathbb{C}^t \oplus \dots$. Thus

$$A_i = A_i^*, \quad B_j = B_j^*, \quad 1 \leq i \leq n, \quad 1 \leq j \leq m.$$

Moreover there exists a unitary $U : \mathcal{H} \rightarrow (\mathbb{C}^t)^\infty$, which induces a faithful $*$ -representation $Ad(U^*)$ of $\mathcal{M}_t(\mathbb{C})^\infty$ on \mathcal{H} satisfying $\rho^\infty = Ad(U)$. Thus by (iii) and (iv), we have

(1)

$$|||P_i(A_1, \dots, A_n)|| - |||P_i(x_1, \dots, x_n)||| < \frac{1}{r}, \quad \forall 1 \leq i \leq r$$

and

$$|||Q_i(B_1, \dots, B_m)|| - |||Q_i(y_1, \dots, y_m)||| < \frac{1}{r}; \quad \forall 1 \leq i \leq r;$$

(2) for $1 \leq k \leq r$, $1 \leq i \leq n$, $1 \leq j \leq m$,

$$\|U^* A_i^\infty U \xi_k - x_i \xi_k\| = \|(U^* A_i^\infty U \xi_k - U^* A_i^\infty U p \xi_k) + (\tilde{x}_i p \xi_k - x_i \xi_k)\| < \frac{2}{r};$$

and

$$\|U^* B_j^\infty U \xi_k - y_j \xi_k\| = \|(U^* B_j^\infty U \xi_k - U^* B_j^\infty U q \xi_k) + (\tilde{y}_j q \xi_k - y_j \xi_k)\| < \frac{2}{r}.$$

This completes the proof of the theorem. \square

Recall the definition of the unital full free product of two unital C^* -algebras as follows.

DEFINITION 5.3. The unital full free product of unital C^* -algebras $\{\mathcal{A}_i\}_{i=1}^m$ ($m \geq 2$) is a unital C^* -algebra \mathcal{D} , denoted by $\mathcal{A}_1 *_{\mathbb{C}} \mathcal{A}_2 *_{\mathbb{C}} \cdots *_{\mathbb{C}} \mathcal{A}_m$, equipped with unital embedding $\{\sigma_i : \mathcal{A}_i \rightarrow \mathcal{D}\}_{i=1}^m$, such that (i) the set $\cup_{i=1}^m \sigma_i(\mathcal{A}_i)$ is norm dense in \mathcal{D} ; and (ii) if ϕ_i is a unital $*$ -homomorphism from \mathcal{A}_i into a unital C^* -algebra $\bar{\mathcal{D}}$ for $i = 1, 2, \dots, m$, then there is a unital $*$ -homomorphism ψ from \mathcal{D} to $\bar{\mathcal{D}}$ satisfying $\phi_i = \psi \circ \sigma_i$, for $i = 1, 2, \dots, m$. If there is no confusion arising, we will identify the element x_i in \mathcal{A}_i with its image in \mathcal{D} .

Suppose that \mathcal{A} and \mathcal{B} are unital C^* -algebras with a family of self-adjoint generators x_1, \dots, x_n , and y_1, \dots, y_m respectively. Let $\mathcal{A} *_{\mathbb{C}} \mathcal{B}$ be the unital full free product of \mathcal{A} and \mathcal{B} , defined as above, with a family of self-adjoint generators $\{x_1, \dots, x_n, y_1, \dots, y_m\}$. Let $\{P_r\}_{r=1}^\infty$, or $\{Q_r\}_{r=1}^\infty$, or $\{\Psi_r\}_{r=1}^\infty$ be the collection of all noncommutative polynomials in $\mathbb{C}\langle X_1, \dots, X_n \rangle$, or $\mathbb{C}\langle Y_1, \dots, Y_m \rangle$, or $\mathbb{C}\langle X_1, \dots, X_n, Y_1, \dots, Y_m \rangle$ respectively, with rational complex coefficients. Then we have the following result.

LEMMA 5.7. Let \mathcal{A}, \mathcal{B} and $x_1, \dots, x_n, y_1, \dots, y_m$ be as above. For any positive integer a , there is a positive integer r so that the following hold:

If $A_1, \dots, A_n, B_1, \dots, B_m$ are self-adjoint elements on a separable Hilbert space \mathcal{K} so that

$$|||P_i(A_1, \dots, A_n)|| - ||P_i(x_1, \dots, x_n)||| < \frac{1}{r}, \quad \forall 1 \leq i \leq r$$

and

$$|||Q_i(B_1, \dots, B_m)|| - ||Q_i(y_1, \dots, y_m)||| < \frac{1}{r}, \quad \forall 1 \leq i \leq r,$$

then, $\forall 1 \leq j \leq a$,

$$\|\Psi_j(A_1, \dots, A_n, B_1, \dots, B_m)\|_{B(\mathcal{K})} < \|\Psi_j(x_1, \dots, x_n, y_1, \dots, y_m)\|_{\mathcal{A} *_{\mathbb{C}} \mathcal{B}} + \frac{1}{a}.$$

PROOF. We will prove the result of the lemma by using contradiction. Assume that there are a positive integer a and a sequence of self-adjoint elements

$$A_1^{(r)}, \dots, A_n^{(r)}, B_1^{(r)}, \dots, B_m^{(r)} \quad \text{in} \quad B(\mathcal{K}^{(r)}), \quad r = 1, 2, \dots$$

satisfying, for all $r \geq 1$,

$$|||P_i(A_1^{(r)}, \dots, A_n^{(r)})|| - ||P_i(x_1, \dots, x_n)||| < \frac{1}{r}, \quad \forall 1 \leq i \leq r \quad (5.18)$$

and

$$|||Q_i(B_1^{(r)}, \dots, B_m^{(r)})|| - ||Q_i(y_1, \dots, y_m)||| < \frac{1}{r}, \quad \forall 1 \leq i \leq r; \quad (5.19)$$

but

$$\max_{1 \leq j \leq a} \{ \|\Psi_j(A_1^{(r)}, \dots, A_n^{(r)}, B_1^{(r)}, \dots, B_m^{(r)})\|_{B(\mathcal{K}^{(r)})} - \|\Psi_j(x_1, \dots, x_n, y_1, \dots, y_m)\|_{\mathcal{A} *_{\mathbb{C}} \mathcal{B}} \} \geq \frac{1}{a}. \quad (5.20)$$

Consider the unital C^* -algebra $\prod_r B(\mathcal{K}^{(r)}) / \sum_r B(\mathcal{K}^{(r)})$ and elements

$$[(A_1^{(r)})_r], \dots, [(A_n^{(r)})_r], [(B_1^{(r)})_r], \dots, [(B_m^{(r)})_r] \in \prod_r B(\mathcal{K}^{(r)}) / \sum_r B(\mathcal{K}^{(r)}).$$

By the inequalities (5.18) and (5.19), there are unital embeddings $\phi_1 : \mathcal{A} \rightarrow \prod_r B(\mathcal{K}^{(r)}) / \sum_r B(\mathcal{K}^{(r)})$ and $\phi_2 : \mathcal{B} \rightarrow \prod_r B(\mathcal{K}^{(r)}) / \sum_r B(\mathcal{K}^{(r)})$ so that

$$\phi_1(x_{i_1}) = [(A_{i_1}^{(r)})_r], \quad \phi_2(y_{i_2}) = [(B_{i_2}^{(r)})_r], \quad 1 \leq i_1 \leq n, \quad 1 \leq i_2 \leq m.$$

By the definition of the full free product $\mathcal{A} *_\mathbb{C} \mathcal{B}$, there is a $*$ -homomorphism $\phi : \mathcal{A} *_\mathbb{C} \mathcal{B} \rightarrow \prod_r B(\mathcal{K}^{(r)}) / \sum_r B(\mathcal{K}^{(r)})$ so that

$$\phi(x_{i_1}) = [(A_{i_1}^{(r)})_r], \quad \phi(y_{i_2}) = [(B_{i_2}^{(r)})_r], \quad 1 \leq i_1 \leq n, \quad 1 \leq i_2 \leq m,$$

where $x_1, \dots, x_n, y_1, \dots, y_m$ are identified as the elements in $\mathcal{A} *_\mathbb{C} \mathcal{B}$. Therefore, $\forall 1 \leq j \leq a$,

$$\begin{aligned} \|\Psi_j([(A_1^{(r)})_r], \dots, [(A_n^{(r)})_r], [(B_1^{(r)})_r], \dots, [(B_m^{(r)})_r])\|_{\prod_r B(\mathcal{K}^{(r)}) / \sum_r B(\mathcal{K}^{(r)})} \\ = \limsup_{r \rightarrow \infty} \|\Psi_j(A_1^{(r)}, \dots, A_n^{(r)}, B_1^{(r)}, \dots, B_m^{(r)})\|_{B(\mathcal{K}^{(r)})} \\ \leq \|\Psi_j(x_1, \dots, x_n, y_1, \dots, y_m)\|_{\mathcal{A} *_\mathbb{C} \mathcal{B}}, \end{aligned}$$

which contradicts the inequality (5.20). □

In [9], Exel and Loring showed that the unital full free product of two residually finite dimensional C^* -algebra is residually finite dimensional, which extends an earlier result by Choi in [6]. In [3], Boca showed that the unital full free product of two quasidiagonal C^* -algebras is also quasidiagonal. Our next result provides the analogue of the preceding results from Choi, Exel and Loring, and Boca in the context of MF algebras.

THEOREM 5.4. *Suppose \mathcal{A} and \mathcal{B} are finitely generated unital MF algebras. Then the unital full free product $\mathcal{A} *_\mathbb{C} \mathcal{B}$ is an MF algebra.*

PROOF. Assume that \mathcal{A} , and \mathcal{B} , are generated by a family of self-adjoint elements x_1, \dots, x_n , and y_1, \dots, y_m respectively. Thus we can assume that $x_1, \dots, x_n, y_1, \dots, y_m$ also generate $\mathcal{A} *_\mathbb{C} \mathcal{B}$ as a C^* -algebra. Assume the $\pi : \mathcal{A} *_\mathbb{C} \mathcal{B} \rightarrow B(\mathcal{H})$ is a faithful $*$ -representation of $\mathcal{A} *_\mathbb{C} \mathcal{B}$ on a separable Hilbert space \mathcal{H} . Let $\{P_r\}_{r=1}^\infty$, or $\{Q_r\}_{r=1}^\infty$, or $\{\Psi_r\}_{r=1}^\infty$ be the collection of all noncommutative polynomials in $\mathbb{C}\langle X_1, \dots, X_n \rangle$, or $\mathbb{C}\langle Y_1, \dots, Y_m \rangle$, or $\mathbb{C}\langle X_1, \dots, X_n, Y_1, \dots, Y_m \rangle$ respectively, with rational complex coefficients.

To show that $\mathcal{A} *_\mathbb{C} \mathcal{B}$ is an MF algebra, it suffices to show that for any positive integer r_0 , there are a positive integer t and self-adjoint matrices

$$A_1, \dots, A_n, B_1, \dots, B_m \in \mathcal{M}_t(\mathbb{C})$$

such that, $\forall 1 \leq j \leq r_0$,

$$\left| \|\Psi_j(A_1, \dots, A_n, B_1, \dots, B_m)\|_{\mathcal{M}_t(\mathbb{C})} - \|\Psi_j(x_1, \dots, x_n, y_1, \dots, y_m)\|_{\mathcal{A} *_\mathbb{C} \mathcal{B}} \right| < \frac{1}{r_0}.$$

Suppose that $\{\xi_r\}$ is a dense subset of the unit ball of \mathcal{H} . Note that \mathcal{A} and \mathcal{B} are MF algebras. For any positive integer r , by Theorem 5.2 we know there are positive integers n_r, m_r ,

and self-adjoint matrices

$$E_1^{(r)}, \dots, E_n^{(r)} \in \mathcal{M}_{n_r}^{s.a.}(\mathbb{C}), \quad F_1^{(r)}, \dots, F_m^{(r)} \in \mathcal{M}_{m_r}^{s.a.}(\mathbb{C}),$$

and unitary operators $U_r : \mathcal{H} \rightarrow (\mathbb{C}^{n_r})^\infty$, $V_r : \mathcal{H} \rightarrow (\mathbb{C}^{m_r})^\infty$ such that

(1)

$$\left| \|P_i(E_1^{(r)}, \dots, E_n^{(r)})\| - \|P_i(x_1, \dots, x_n)\| \right| < \frac{1}{2r}, \quad \forall 1 \leq i \leq r; \quad (5.21)$$

$$\left| \|Q_i(F_1^{(r)}, \dots, F_m^{(r)})\| - \|Q_i(y_1, \dots, y_m)\| \right| < \frac{1}{2r}, \quad \forall 1 \leq i \leq r. \quad (5.22)$$

(2)

$$\|U_r^* (E_{i_1}^{(r)})^\infty U_r \xi_j - \pi(x_{i_1}) \xi_j\| < \frac{1}{2r}, \quad \forall 1 \leq i_1 \leq n, \quad 1 \leq j \leq r; \quad (5.23)$$

$$\|V_r^* (F_{i_2}^{(r)})^\infty V_r \xi_j - \pi(y_{i_2}) \xi_j\| < \frac{1}{2r}, \quad \forall 1 \leq i_2 \leq m, \quad 1 \leq j \leq r. \quad (5.24)$$

Note both $\{U_r^* (E_{i_1}^{(r)})^\infty U_r\}_{i_1=1}^n$ and $\{V_r^* (F_{i_2}^{(r)})^\infty V_r\}_{i_2=1}^m$ are sets of quasidiagonal operators on the Hilbert space \mathcal{H} . By Theorem 5.3, there are a positive integer t_r , families of self-adjoint matrices

$$\{A_1^{(r)}, \dots, A_n^{(r)}\} \subset \mathcal{M}_{t_r}^{s.a.}(\mathbb{C}) \quad \{B_1^{(r)}, \dots, B_m^{(r)}\} \subset \mathcal{M}_{t_r}^{s.a.}(\mathbb{C}), \quad (5.25)$$

and unitary $W_r : \mathcal{H} \rightarrow (\mathbb{C}^{t_r})^\infty$ such that

(3)

$$\left| \|P_i(A_1^{(r)}, \dots, A_n^{(r)})\| - \|P_i(E_1^{(r)}, \dots, E_n^{(r)})\| \right| < \frac{1}{2r}, \quad \forall 1 \leq i \leq r, \quad (5.26)$$

and

$$\left| \|Q_i(B_1^{(r)}, \dots, B_m^{(r)})\| - \|Q_i(F_1^{(r)}, \dots, F_m^{(r)})\| \right| < \frac{1}{2r}, \quad \forall 1 \leq i \leq r; \quad (5.27)$$

(4)

$$\|W_r^* (A_{i_1}^{(r)})^\infty W_r \xi_j - U_r^* (E_{i_1}^{(r)})^\infty U_r \xi_j\| < \frac{1}{2r}, \quad \forall 1 \leq i_1 \leq n, \quad 1 \leq j \leq r; \quad (5.28)$$

and

$$\|W_r^* (B_{i_2}^{(r)})^\infty W_r \xi_j - V_r^* (F_{i_2}^{(r)})^\infty V_r \xi_j\| < \frac{1}{2r}, \quad \forall 1 \leq i_2 \leq m, \quad 1 \leq j \leq r. \quad (5.29)$$

Combining the inequalities (5.21), (5.22), (5.16) and (5.27), we have

$$\left| \|P_i(A_1^{(r)}, \dots, A_n^{(r)})\| - \|P_i(x_1, \dots, x_n)\| \right| < \frac{1}{r}, \quad \forall 1 \leq i \leq r; \quad (5.30)$$

$$\left| \|Q_i(B_1^{(r)}, \dots, B_m^{(r)})\| - \|Q_i(y_1, \dots, y_m)\| \right| < \frac{1}{r}, \quad \forall 1 \leq i \leq r. \quad (5.31)$$

Combining the inequalities (5.23), (5.24), (5.28) and (5.29), we have

$$\|W_r^*(A_{i_1}^{(r)})^\infty W_r \xi_j - \pi(x_{i_1}) \xi_j\| < \frac{1}{r}, \quad \forall 1 \leq i_1 \leq n, 1 \leq j \leq r; \quad (5.32)$$

$$\|W_r^*(B_{i_2}^{(r)})^\infty W_r \xi_j - \pi(y_{i_2}) \xi_j\| < \frac{1}{r}, \quad \forall 1 \leq i_2 \leq m, 1 \leq j \leq r. \quad (5.33)$$

By Lemma 5.7 and the inequalities (5.30), (5.31), we know that, for $1 \leq j \leq r_0$,

$$\limsup_{r \rightarrow \infty} \|\Psi_j(A_1^{(r)}, \dots, A_n^{(r)}, B_1^{(r)}, \dots, B_m^{(r)})\|_{\mathcal{M}_{t_r}(\mathbb{C})} \leq \|\Psi_j(x_1, \dots, x_n, y_1, \dots, y_m)\|_{\mathcal{A} *_{\mathbb{C}} \mathcal{B}}. \quad (5.34)$$

Note $\{W_r^*(A_{i_1}^{(r)})^\infty W_r, W_r^*(B_{i_2}^{(r)})^\infty W_r\}_{r, i_1, i_2}$ is a bounded subset in $B(\mathcal{H})$. Since $\{\xi_r\}_{r=1}^\infty$ is a dense subset of the unit ball of \mathcal{H} , by the inequality (5.32) and (5.33) we know that, as r goes to infinity,

$$W_r^*(A_{i_1}^{(r)})^\infty W_r \rightarrow \pi(x_{i_1}), \quad W_r^*(B_{i_2}^{(r)})^\infty W_r \rightarrow \pi(y_{i_2}) \text{ in } SOT, \quad \forall 1 \leq i_1 \leq n, 1 \leq i_2 \leq m.$$

Hence, for $1 \leq j \leq r_0$

$$\begin{aligned} \liminf_{r \rightarrow \infty} \|\Psi_j(A_1^{(r)}, \dots, A_n^{(r)}, B_1^{(r)}, \dots, B_m^{(r)})\|_{\mathcal{M}_{t_r}(\mathbb{C})} \\ = \liminf_{r \rightarrow \infty} \|\Psi_j(W_r^*(A_1^{(r)})^\infty W_r, \dots, W_r^*(A_n^{(r)})^\infty W_r, W_r^*(B_1^{(r)})^\infty W_r, \dots, W_r^*(B_m^{(r)})^\infty W_r)\|_{B(\mathcal{H})} \\ \geq \|\Psi_j(x_1, \dots, x_m, y_1, \dots, y_m)\|_{\mathcal{A} *_{\mathbb{C}} \mathcal{B}}. \end{aligned} \quad (5.35)$$

From the fact (5.25) and inequalities (5.34) and (5.35), it follows that, for the given r_0 , there are a positive integer t and self-adjoint matrices

$$A_1, \dots, A_n, B_1, \dots, B_m \in \mathcal{M}_t(\mathbb{C})$$

such that, $\forall 1 \leq j \leq r_0$,

$$\left| \|\Psi_j(A_1, \dots, A_n, B_1, \dots, B_m)\|_{\mathcal{M}_t(\mathbb{C})} - \|\Psi_j(x_1, \dots, x_n, y_1, \dots, y_m)\|_{\mathcal{A} *_{\mathbb{C}} \mathcal{B}} \right| < \frac{1}{r_0}.$$

This completes the proof of the theorem. \square

REMARK 5.1. Using the similar argument in the proof of Theorem 5.3, the result of Theorem 5.3 can be quickly extended as follows: *Suppose \mathcal{A} and \mathcal{B} are separable unital MF algebras. Then the unital full free product $\mathcal{A} *_{\mathbb{C}} \mathcal{B}$ is an MF algebra.*

In [15], Haagerup and Thorbjørnsen showed $C_r^*(F_2)$ is an MF algebra. Combining with Voiculescu's discussion in [32], they were able to conclude a striking result that $\text{Ext}(C_r^*(F_2))$ is not a group. Also based on [32], Brown showed in [4] that if \mathcal{A} is an MF algebra and $\text{Ext}(\mathcal{A})$ is a group, then \mathcal{A} is a quasidiagonal C^* -algebra. It is a well-known fact that $C_r^*(F_2)$ is not a quasidiagonal C^* -algebra and any subalgebra of a quasidiagonal C^* -algebra is again quasidiagonal. Now from Haagerup and Thorbjørnsen's result on $C_r^*(F_2)$ and our Theorem 5.4, it quickly follows the next corollary.

COROLLARY 5.3. *Suppose that \mathcal{B} is a unital separable MF algebra. Then $C_r^*(F_2) *_{\mathbb{C}} \mathcal{B}$ is an MF algebra. Moreover, $\text{Ext}(C_r^*(F_2) *_{\mathbb{C}} \mathcal{B})$ is not a group.*

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Don Hadwin; Qihui Li; Junhao Shen

Department of Mathematics and Statistics

University of New Hampshire

Durham, NH, 03824

Email: don@math.unh.edu; qme2@cisunix.unh.edu; jog2@cisunix.unh.edu